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# MAXIMUM ENTROPY SPECTRAL ANALYSIS

G H ROSS

Geophysical Fluid Dynamics Laboratory
Meteorological Office
Bracknell, Berkshire

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The Maximum Entropy Method uses Information Theory Concepts to solve the problem of Autocorrelation Extension for a finite length of a time series. The Spectral Estimate is most useful for short length data, where the frequency resolution is superior to other methods. Numerical Experiments have been conducted on synthetic time series using a Meteorological Office computer routine.

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## Power Spectral Analysis using Burg's Algorithm

## The Maximum Entropy Method. MEM

The MEM, also known as MESA Maximum Entropy Spectral Analysis, is a 'data adaptive' method for producing a Power Spectrum from a discrete time series. It was developed by J P Burg (1) and was also proposed by Parzen (2) who derived it from the theory of autoregressive (AR) modelling.

MEM is 'data adaptive' (3) in that the window function is not defined as in the cases of Blackman Tukey methods (4) or the Periodogram or Fast Fourier Transform methods (5,6), but is implicitly altered by the data being processed to suit the noise characteristics of the signal. In this way it is like the Maximum Likelihood Method (7,8) (MLM), which in fact can be shown to be derivable from MEM (9).

It is exactly equivalent to modelling by autoregressive decomposition (10,11) and suffers the same problem of the best choice of the AR order.

The Maximum Entropy concept comes from an Information Theory approach but an equivalent description, providing further insight into the working of the method comes from Filter Theory. This description is an alternative form of the AR models.

The spectral Estimate produced by MEM has a higher variance than most other methods, but this is more than compensated by a significantly higher resolution and the capability of resolving wavelengths comparable to the length of the data sample (2,12,13).

Information Theory Origins (14) Truncation of a time series.

The basis for the Information Theory ideas develops when an attempt is made to determine the Power Spectrum from a truncated time series.

Let $x_i = x(t_i) : \; i = 1, 2, \cdots N. \qquad t_i - t_{i-1} = \Delta t$ 1)

be a discrete, equally-spaced time series of zero mean, which is a segment of an infinite stationary time series.

The autocovariances are

$$\phi_k = \phi(k) = E\left[ x_j x_{j+k} \right] = \phi_{-k} \qquad 2)$$

autocorrelations are covariances normalised with $\phi(0) = 1$ where $E$ is the expectation or expected value.

An estimate for $\phi(k)$ , for the truncated series might be:-

$$\phi(k) = \frac{1}{(N-k)} \sum_{j=1}^{N-k} x_j x_{j+k} \quad : \quad k < N \qquad 3)$$

The Power Spectral Density is the Fourier Transform of the autocovariances:-

$$S(f) = \Delta t \sum_{-\infty}^{\infty} \phi(k) \exp(-2\pi i f k \Delta t) \qquad 4)$$

and this might be approximated:-

$$S_N(f) = \Delta t \sum_{-N+1}^{N-1} \phi(k) \exp(-2\pi i f k \Delta t) \qquad 5)$$

This is equivalent to 4) with $\phi(k) = 0 \; ; |k| \leq N$. $S_N(f)$ is limited in applicability as an estimator for $S(f)$, since for short length data, or for wavelengths long compared to the length of the data, $S_N(f)$ may be badly in error.

All forms of window fitting which set the autocorrelations to zero outside the data range are in effect putting information into the data which does not exist.

It is desirable to find a method of autocorrelation extension which does not have this bias.

Maximisation of Entropy

A method which uses only the available data would be best. Such a method would recognise the lack of information about the time series outside the truncation length. Loss of Information is gain of Entropy, and a consistent

measure of Entropy Density is

$$h = \frac{1}{2}\ln(f_N) + \frac{1}{4f_N}\int_{-f_N}^{f_N} \ln(S(f))\,df. \qquad 6)$$

$f_N$ is the Nyquist frequency = $\frac{1}{2}/\Delta t$, and is the highest frequency resolvable by the time series.

Putting 4) into 6) and ignoring the constant

$$h = \frac{1}{4f_N}\int_{-f_N}^{f_N} \ln\left[\Delta t \sum_{-\infty}^{\infty} \phi(k)\exp(-2\pi i k f/\Delta t)\right] df. \qquad 7)$$

If this is maximised with respect to the unknown autocorrelations it is equivalent to maximising the ignorance about them and minimising the information.

$$\frac{\partial h}{\partial \phi(k)} = 0 \quad |k| \geq N$$

$$= \int_{-f_N}^{f_N} \frac{\exp(-2\pi i f k \Delta t)}{S(f)}\,df .$$

This must mean that the fourier series for $1/S(f)$ truncates

ie $$\frac{1}{S(f)} = \frac{1}{2f_N}\sum_{-N+1}^{N-1} C_k \exp(-2\pi i f k \Delta t) .$$

with $C_k = C_{-k}^*$ to ensure that $1/S(f)$ is real, (* is the complex conjugate).

$$S_h(f) = \frac{2f_N}{\sum_{-N+1}^{N-1} C_k \exp(-2\pi i f k \Delta t)} \qquad 8)$$

$C_k$ must be chosen so that $S_h(f)$ gives the known autocorrelations

$$\phi(k) = \int_{-f_N}^{f_N} S_h(f)\exp(2\pi i f k \Delta t)\,df \;:\; |k| < N. \qquad 9)$$

since the autocorrelations are the inverse transform of the Power Spectrum.

Equation 8) is used by Smylie et al (14) who show the analysis used to derive equation 23).

An alternative formulation using Lagrangian multipliers of the constraints eq 9) to define a variational principle is used Edward and Fitelson 15)

Comparing 14) with 8). It is seen that they must be equivalent and all that is needed to find the Maximum Entropy Spectral Estimate are the weights $\gamma_k$ of the filter which 'whitens' the time series.

Autoregression and Prediction

An Autoregressive Process of order $M$ is defined as (16)

$$x(t_j) = \sum_{j=1}^{M} a_i x(t_{j-i}) + Z_j \qquad 15)$$

where $a_i$ are the AR components

$Z_j$ are zero mean, randomly distributed series with variance $\sigma_z^2$; also called the innovation.

In an AR process, previous values determine the present value. Because of this 'memory' of the past, it has been called a Markov process. The AR process can be shown to be the discrete version of a linear differential equation with a random impulse forcing function.

A similar filter which is used to predict values of a series at some subsequent point from previous values is called a Prediction Filter (18). The value of the series at $\alpha$ units in the future

$$\hat{y}_{j+\alpha} = \sum_{k=1}^{M} y_{j-k} g_k^{\alpha} \qquad 16)$$

where $g_k^{\alpha}$ are the $\alpha$ length prediction filter weights.

The difference between the correct value and the estimate

$$y_{j+\alpha} - \hat{y}_{j+\alpha} = \sum_{k=0}^{M+\alpha} y_{j+\alpha-k} \gamma_k^{\alpha} = \varepsilon_{j+\alpha} \qquad 17)$$

$$17)$$

$\gamma_k^{\alpha}$ are the $\alpha$ length Prediction Error Filter weights (PEF)

$$\{\gamma_k^{\alpha}\} = \{1, 0, 0 \cdots \alpha \text{ ZEROES} \cdots, -g_1^{\alpha}, -g_2^{\alpha} \cdots\} \qquad (17, 13, 18, 19)$$

$$\delta \int_{-f_N}^{f_N} df \left[ \ln(S(f)) - \sum_{-N+1}^{N-1} \lambda_k \left\{ S(f) \exp(2\pi i f k \Delta t) - \frac{\phi(k)}{2 f_N} \right\} \right] = 0 \qquad 10)$$

which gives $$\frac{1}{S(f)} = \sum_{-N+1}^{N-1} \lambda_k z^k \, .$$

$$S(f) = \frac{1}{\sum_{-N+1}^{N-1} \lambda_k z^k} \quad : \quad z = \exp(2\pi i f k \Delta t) \qquad 11)$$

which is identical to 8).

Filter Theory Origin (16,17)

A process $H$ has a set of input data $X$ and output data $Y$. If some knowledge is needed of the output produced, given other input, then the process may be modelled by a linear filter $G$ acting on the data $X$ to produce an output estimate $\hat{Y}$.

$$Y = H(X)$$

$$\hat{y}_j = \sum_i g_i x_{j-i} \qquad 12)$$

where $g_i$ are the filter weights.

Consider a filter $\Gamma$ operating on a time series $X$ which removes all the deterministic signal leaving only random noise. $z_j$

$$z_j = \sum_i \gamma_i x_{j-i} \qquad 13)$$

Taking the Fourier Transform of 13);

$$Z(\omega) = \Gamma(\omega) X(\omega) = K \text{ constant.} \qquad \omega = 2\pi f$$

since the FT of 'white noise' is a constant. $\Gamma(\omega)$ is the Transfer Function of the filter.

The square of the modulus of the Fourier Transform of a time series is a measure of the Power Spectrum

$$|X(\omega)|^2 = \frac{K^2}{|\Gamma(\omega)|^2} \qquad 14)$$

$\gamma'_k$ is the unit length $M+1$ $k$ order PEF.

It can be shown that the general PEF $\gamma^{\alpha}_k$ can be derived from $\gamma'_k$ (17,14) and so equations need only use the unit length filter.

Using a prediction filter assumes that the time series can be modelled by an AR series.

Least Squares Estimate

The power in the error series is the convolution of the filter and the original time series

$$P_{MM} = E[\varepsilon_j^2] \qquad 18)$$

Suppose that, as in equation 9, a process $H$ can be modelled by a filter $G$. The filter elements must be chosen to give a close approximation to the output of $H$. If $g_i$ are chosen such that the mean square error between $Y$ and $Y'$ is minimised, the filter $G$ is called a Wiener Filter (15, 18) and the condition to be observed is the Wiener-Hopf Equation

$$\Phi_{YX}(j) = \sum_{k=-M}^{M} g_k \Phi_{XX}(j-k) \qquad 19)$$

where $\Phi_{YX}$ is the correlation between the output and the input time series

$\Phi_{XX}$ is the autocorrelation of the input series

If $Y$ is the same series as $X$, $g_k$ is the prediction filter and the noise Power

$$P_{M+1} = \sum_{k=0}^{M} \gamma_k \Phi_{XX}(-k) \qquad 20)$$

The Wiener Filter form for the prediction filter is

$$\begin{vmatrix} \phi(1) \\ \phi(2) \\ \vdots \\ \phi(M) \end{vmatrix} = \begin{vmatrix} \phi(0) & \phi(1) & \cdots & \phi(1-M) \\ \phi(1) & \phi(0) & \cdots & \phi(2-M) \\ \vdots & \vdots & & \vdots \\ \phi(M-1) & \phi(M-2) & \cdots & \phi(0) \end{vmatrix} \begin{vmatrix} g_1 \\ g_2 \\ \vdots \\ g_M \end{vmatrix} \qquad 21)$$

Using the noise power equation 20) in this system of simultaneous equations

$$\begin{vmatrix} -\phi(0) & \phi(-1)\ \phi(-2) & \cdots & \phi(-n) \\ -\phi(1) & \phi(0)\ \phi(-1) & \cdots & \phi(1-n) \\ -\phi(2) & \phi(1)\ \phi(0) & \cdots & \phi(2-n) \\ \vdots & \vdots\ \vdots & & \vdots \\ -\phi(n) & \phi(n-1)\ \phi(n-2) & \cdots & \phi(0) \end{vmatrix} \begin{vmatrix} -1 \\ g_1 \\ g_2 \\ \\ g_n \end{vmatrix} = \begin{vmatrix} -P_{n+1} \\ \phi(1)-\phi(1) \\ \phi(2)-\phi(2) \\ \vdots \\ \phi(n)-\phi(n) \end{vmatrix}$$

which is

$$\begin{vmatrix} P_{n+1} \\ 0 \\ \vdots \\ 0 \end{vmatrix} = \begin{vmatrix} \phi(0)\ \phi(1) & \cdots & \phi(n) \\ \phi(1)\ \phi(0) & \cdots & \phi(n-1) \\ \vdots\ \vdots & & \vdots \\ \phi(n)\ \phi(n-1) & \cdots & \phi(0) \end{vmatrix} \begin{vmatrix} \gamma_0 \\ \gamma_1 \\ \vdots \\ \gamma_n \end{vmatrix} \qquad 22)$$

The Matrix $\Phi_{kj} = \phi(k-j)$ is the correlation matrix and has the symmetries

Toeplitz - equidiagonal $\phi_{11} = \phi_{22} = \cdots$

Hermitian symmetrical about diagonal

writing 19) in Tensor notation

$$P_{n+1}\,\delta_{k,0} = \phi_{kj}\,\gamma_j$$

$\delta_{kj}$ is the Kronecker function $= 1 \quad k=j \;;\; = 0 \quad k \neq j$

the $\gamma_j$ can be identified as the whitening filter and equation 22) can be inverted to give

$$\underline{\gamma} = \underline{\underline{\Phi}}^{-1}\,\underline{P}$$

Smylie et al use the prediction error filter and derive from equation 8) the following relation

$$S_x(f) = \frac{\Delta t\, P_{n+1}}{\left| \sum_{k=0}^{n} \gamma_k \exp(-2\pi i f k \Delta t) \right|} \qquad 23)$$

Burg's Maximum Entropy Method

Normally a square matrix inversion needs $N^3$ operations, however, because of the symmetries, recursive methods, due to Levinson and Durbin (reviewed by Wiggins and Robinson (20)) cuts down the number of operations to $N(N+1)$ . A normal Fourier Transform requires $N^2$ operations, while a Fast Fourier Transform takes $N \log_2(N)$ .

However this still requires an estimation of the autocorrelations $\phi(k)$ and this can bring bias back into the estimate of the Spectrum.

Burg, in papers presented to the Society of Exploration Geophysics, introduced a formal procedure which eliminates this bias. The method makes an estimate of the $M$th order filter from the $M-1$th, using a minimisation of the Power, with respect to the $M$th component of the $M$th order to complete the set of equations.

$P_M$, the error power of the $M$ order (eq 20) starts at $P_1$. This is taken as the autocorrelation at zero'th and

$$P_1 = \frac{1}{N} \sum_{j=1}^{N} x_j^2 = \phi(0). \qquad 24)$$

is the estimate.

The form of the filter at the First Order (eq 22) is

$$\begin{matrix} \phi(0) & \phi(1) & \gamma_{0,1} & P_2 \\ \phi(1) & \phi(0) & \gamma_{1,1} & 0 \end{matrix} \qquad 25)$$

$\gamma_{B,A}$ signifies the $B$th component of the $A$th order

$\gamma_{0,A}$ is always unity (eq 22)

The filter $\gamma$ can be used on the time series in both directions, forward and backward, and this leads to the definition of a forward and backward error series:

$$\begin{aligned} \varepsilon^f_{j,1} &= f_j + \gamma_{1,1} f_{j+1} \\ \varepsilon^b_{j,1} &= f_j + \gamma_{1,1} f_{j-1} \end{aligned} \qquad 26)$$

The Noise power is taken as the average of the forward and backward mean square errors

$$\begin{aligned} P_2 &= \frac{1}{2}\left(P_2^f + P_2^b\right) = \frac{1}{2(N-1)} \sum_{j=1}^{N-1} \left[ \left(\varepsilon^f_{j,1}\right)^2 + \left(\varepsilon^b_{j,1}\right)^2 \right] \\ &= \frac{1}{2(N-1)} \sum_{j=1}^{N-1} \left[ \left(x_j + \gamma_{1,1} x_{j+1}\right)^2 + \left(x_j + \gamma_{1,1} x_{j+1}\right)^2 \right] \end{aligned} \qquad 27)$$

$\gamma_{11}$ is chosen to minimise $P_2$ and this gives

$$\gamma_{11} = \frac{-2\sum_{j=1}^{N-1}(x_j x_{j+1})}{\sum_{j=1}^{N-1}(x_j^2 + x_{j+1}^2)} \qquad 28)$$

Equation 25) is solved for $\phi(1)$ and $P_2$ is rewritten in terms of $\gamma_{11}$

$$\phi(1) = -\gamma_{11}\phi(0) \quad ; \quad P_2 = (1-\gamma_{11}^2)P_1$$

The next step involves 22) in the form

$$\begin{matrix}\phi(0) & \phi(1) & \phi(2) \\ \phi(1) & \phi(0) & \phi(1) \\ \phi(2) & \phi(1) & \phi(0)\end{matrix} \quad \begin{matrix}\gamma_{02} \\ \gamma_{12} \\ \gamma_{22}\end{matrix} = \begin{matrix}P_3 \\ 0 \\ 0\end{matrix} \qquad 29)$$

$\gamma_{02} = 1$ Therefore there are 4 unknowns $P_3$, $\gamma_{12}$, $\gamma_{22}$, $\phi(2)$; and 3 equations.

An estimate for $P_3$ is made in the same way as $P_2$ and is minimised wrt $\gamma_{22}$

$$P_3 = \frac{1}{2(N-2)}\sum_{j=1}^{N-2}\left[(\varepsilon_{j,2}^f)^2 + (\varepsilon_{j,2}^b)^2\right]$$

$$\frac{\partial P_3}{\partial \gamma_{22}} = 0 \qquad 30)$$

These equations are rearranged such that it is $\gamma_{22}$ which is estimated from the data and the previous filter

$$\gamma_{22} = \frac{-2\sum_{j=1}^{N-2}(x_{j+2} + \gamma_{11}x_{j+1})(x_j + \gamma_{11}x_{j+1})}{\sum_{j=1}^{N-2}\left[(x_{j+2} + \gamma_{11}x_{j+1})^2 + (x_j + \gamma_{11}x_{j+1})^2\right]} \qquad 31)$$

$$= \frac{-2\sum_{j=1}^{N-2}(\varepsilon_{j+2,1}^f)(\varepsilon_{j,1}^b)}{\sum_{j=1}^{N-2}\left[(\varepsilon_{j+2,1}^f)^2 + (\varepsilon_{j,1}^b)^2\right]}$$

and the other three unknowns

$$\gamma_{12} = \gamma_{11} + \gamma_{22}\gamma_{11}$$

$$\phi(2) = -\gamma_{12}\phi(1) - \gamma_{22}\phi(0)$$

$$P_3 = (1-\gamma_{22}^2)P_2$$

This set of equations can be generalised for the $M^{th}$ recursion

$$\gamma_{MM} = \frac{-2 \sum_{j=1}^{N-M} (\varepsilon^f_{j+M,M-1})(\varepsilon^b_{j,M-1})}{\sum_{j=1}^{N-M-1} \left[ (\varepsilon^f_{j+M,M-1})^2 + (\varepsilon^b_{j,M-1})^2 \right]} \quad a)$$

$$P_{M+1} = (1-\gamma_{MM}) P_M \quad b) \quad 32)$$

$$\varepsilon^f_{j,M} = x_j + \gamma_{1M} x_{j-1} + \gamma_{2M} x_{j-2} \cdots + \gamma_{MM} x_{j-M} \quad c)$$

$$\varepsilon^b_{j,M} = x_j + \gamma_{1M} x_{j+1} + \gamma_{2M} x_{j+2} \cdots + \gamma_{MM} x_{j+M} \quad d)$$

$$\gamma_{j,M} = \gamma_{j,M-1} + \gamma_{MM} \gamma_{M-j,M-1} \quad e)$$

$$\gamma_{0M} = 1 \quad f)$$

$$\phi(M) = -\sum_{j=1}^{M} \gamma_{j,M} \phi(M-j) \quad g)$$

So, using only the minimisation of the mean square power of the filter output, estimates of the Power Spectrum and the autocorrelations can be found in a consistent way, instead of making assumptions for all the autocorrelations, as do the window-fitting methods, both within the range of the data and by the zero-extension approximation.

Akaike Criterion

There is a catch. As in window methods, the lag to be chosen for the 'best estimate' is a free parameter. In Window processing, a rule of thumb is to take the lag at which the spectrum is just stable. This point can be unclear, however, and MEM does not possess a criterion for stability. A second method is to use a value of $M = N/3$. Returning to the origins of MEM, the theoretical work on the statistics of autoregressive decomposition is applicable to the estimate of the order.

Akaike has introduced a statistic called the Final Prediction Error (21 ). FPE. FPE is defined as the mean square prediction error

$$FPE = E\left[(x_i - \hat{x}_i)^2\right] = E\left[z_i^2\right] \qquad 33)$$

This is not quite the same as $P_{M+1}$ in equation 18).

The reasoning comes from theoretical statistics. Equation 15) is a process which generates the values of $x(t_i)$. Equation 16) is a filter which predicts values for $\hat{y}(t_i)$. For a fixed set of data, the filter in equation 10) will produce a specific set of filter weights by the MEM recursion at each order. For a different set of the same data, eg. a different section of a long time series, a different set of filter weights will be produced. The difference between the two sets is a result of statistical fluctuations. An AR process can have infinitely many realisations, all statistically equivalent.

The difference between the mean square error of equation 18) and the theoretical concept of 33) is that 33) contains a definition of the unresolvable statistical deviation from the true autoregressive components. This is not constant with order. Consider a series generated by an AR process of order $M$. By the Burg method, orders greater than $M$ can be obtained by recursion. As the order is increased above $M$, the statistical deviation of the components, found for any one realisation, from the true components becomes larger. The error power $P_{M+1}$ must become smaller as the order is increased (Equation 32b)) and Akaike proposed that, if an estimate for FPE could be found, the order at which it became a minimum would also be the best estimate of the true order. In a series of papers, Akaike laid the theoretical basis for taking the FPE as a factor of $P_M$. He found that the best estimate of FPE at order was

$$FPE(M) = \frac{N+M+1}{N-M+1} \cdot P_M \qquad 34)$$

for an error series with zero mean, and

$$FPE(M) = \frac{N+M}{N-M} \cdot P_M \qquad 35)$$

if the mean had not previously been removed.

The estimate of the error Power $P_M$ is produced by fitting the filter to the data. The value of FPE is found by fitting the filter to the process which has produced the data.

Results of Numerical Experiments

Gersche and Sharp (10) carried out a numerical experiment on the fitting of the order of an AR process by the FPE criterion. Using a finite length autoregression-moving average (ARMA) process, equivalent to an infinite AR process, they produced realisations of various lengths. They found that the estimated order increased with the number of data points in a pseudo-logarithmic fashion. They used a method called the Yule-Walker (YW) method to estimate the components and found that the spectrum produced according to the FPE criterion fitted the true spectrum very closely over a 50 db range of spectral power.

Ulrych and Bishop (11) reviewed the theory of AR processes and the relation to the MEM spectrum and extended the numerical experiments in a comparison of the YW and Burg estimates. They found that the Burg method had a consistently higher variance compared with YW, which uses autocorrelation estimation in calculating the AR components. However YW had a power resolution, nearly the same as a periodogram estimate. Also, when the data was highly harmonic, ie, the spectrum was 'peaky', the FPE criterion using the YW method consistently produced a lower order than the Burg method, the Burg order being a better estimate for the spectral order.

As Lacoss has pointed out, the MEM estimate is an estimate of the power spectral density and therefore the area under the curve is the estimate of the signal power.

Lacoss (3) compared estimates produced by a window-fitting method, the Bartlett method (4), the Maximum Likelihood Method (7) and MEM.

The conventional window estimate is

$$P_W = \frac{1}{2M-1} \sum_{-M+1}^{M-1} \omega_k \phi(k) z^k \qquad 36)$$

where $\omega_k$ are the weights of the window function

$$W = \frac{1}{2\pi i} \oint \sum_{-M+1}^{M-1} c_k z^k \qquad 37)$$

in a continuous form

$$W = \frac{1}{2T} \int_{-T}^{T} \omega(t) \exp(2\pi i f t)\, dt$$

where $T = M\Delta t$ is the record length.

So the power spectrum is the convolution of the window function and the true power spectrum, the Fourier transform of the autocorrelations. The window function has a wide central peak and may have secondary side lobes. Thus a trad-off between resolution and the generation of power at spurious frequencies is made by a suitable choice of weights. Jenkins and Watt call this 'window carpentry'.

When the window weights are chosen in the shape of a triangle, this is the Bartlett window

$$\omega(t) = \frac{1}{T}\left(1 - \frac{|t|}{T}\right) \quad ; \quad 0 \leq |t| \leq T$$
$$= 0 \quad ; \quad |t| > T.$$

The Maximum Likelihood Mathod was introduced by Capon (7) to analyse the time series from a two-dimensional seismic array. In the one-dimensional form Lacoss showed that the Power Estimate at frequency was equivalent to the power passed by a filter with Unity Transfer function at that frequency and which suppresses all other frequencies.

Lacoss analysed the three methods by using a Taylor expansion of the Spectral Estimate obtained from an exact correlation function representing a line frequency $f_0$

$$\rho_k = \delta_{k0} + \alpha \cos(2\pi f_0 k \Delta t) \qquad 38)$$

$\delta_{k0}$ is the Kronecker $\delta$ function.

His results are shown in Table 1 below.

| Estimate | Bandwidth Hz | Peak Value | Peak Bandwidth Product |
|---|---|---|---|
| Bartlett | $\sqrt{6}/(\pi\Delta t M)$ | $\alpha/2$ | $\sqrt{6}\alpha/(2\pi\Delta t M)$ |
| Max Likelihood Method MLM | $2\sqrt{3}/(\pi\Delta t M\sqrt{\alpha\pi})$ | $\alpha/2$ | $\sqrt{6\alpha\pi}/(\pi\Delta t M)$ |
| MEM | $4/(\pi\Delta M^2\alpha)$ | $\alpha^2 N^2\Delta t/4$ | $\alpha/\pi$ |

The peak value is the statistic for the power in both the Bartlett and MLM, and the peak bandwidth product, or the area under the peak is the correct power for MEM.

The comparison of the resolutions show that the Bartlett Bandwidth varies as the reciprocal of the lag: $\frac{1}{M}$ . The MLM goes as $\frac{1}{M^{3/2}}$ and MEM as $\frac{1}{M^2}$ .

Lacoss surmises that, when two peaks are adjacent, the curve distorts such that the area remains constant, and his numerical experiments do show this.

$\alpha$ can also be recovered from the peak value, but considering the form of equation 23) the peak value occurs near a zero of the denominator, i.e. near a zero of

$$\sum_{k=0}^{M} \gamma_k z^k \qquad z = \exp(2\pi i f \Delta t)$$

or conversely, near a pole of

$$\frac{1}{\sum_{k=0}^{M} \gamma_k z^k} = \omega(f e^{i\theta})$$

The function $\omega$ cannot have a pole within the unit circle (14) and so peaks occur when the poles lie near the unit circle. The procedure to evaluate the $\gamma_k$ is very sensitive to round off error when this happens, and this makes it difficult to find the positions of the poles to the accuracy needed. Consequently the peak value of $S(f)$ and the bandwidth of the peak have a higher random fluctuation. The area is proportional to the Peak-Bandwidth product, and represents the total power in the peak. It has a smaller variance, since the deviations of the peak and width partly cancel. The uncertainty of the pole position accounts for the small frequency shifts as reported by Ulrych (12).

The difference in the resolution of the MLM and MEM has been explained

by Burg (9), who found that the MLM estimate is an 'averaged' value of all the MEM estimates up to order M. The averaging takes the form of 'parallel resistor network averaging'. The reciprocal of the MLM value is equal to the sums of the reciprocals of the MEM values.

$$\frac{1}{S_L(f)} = \sum_{m=1}^{M} \frac{1}{S_H(f,m)}$$

Ulrych compared a short segment of a sinusoidal wave analysed by MEM and by a Fourier Transform method. He found that changing the phase of the segment caused frequency shifts in both cases, although the width of the peak was much larger for the FT estimate. He showed that, in one example, when the record length was less than 0.58 of the wave period, then the MEM still gave a single sharp peak, as opposed to the FT estimate, which gave the frequency as zero. Toman (22) has shown that the behaviour of the Fourier Transform to waves of length comparable to the length of the data is considerably dependant on the window function broadening of the line frequency interfering with the negative frequency lobe of the spectrum. He found that the frequency estimate was dependant on the phase of the sample and the ratio of the wavelength to the record length. When the ratio was less than 0.58, the frequency estimate was zero.

Chen and Stogen extended Ulrych's work and showed that, although the resolution was much better with MEM, the particular example that Ulrych took was rather fortuitous, since his choice of data and spectral order gave the correct result by accident. They found that high signal to noise ratios gave spurious peaks at the same power as the hidden signal when too high a spectral order was taken.

Variance of MEM

The Maximum Entropy Method does not yet have a consistent statistical test for the evaluation of significance. Blackman and Tukey (4) define a number known as the equivalent degrees of freedom k

$$k = 2 \frac{(\text{Average Power})^2}{(\text{Variance of Power})}$$

when the power spectral estimate is assumed to have a $\chi^2$ distribution. They show that for a Bartlett window, and for the Hamming and Hanning windows

$$k \quad 2N/_M \text{ or } 2\,L/_T$$

where N(L) is the length of the data

N(T) is the length of the window

This is for a slowly varying function P(f) the Fourier amplitude spectrum. When P(f) is not smooth Blackman and Tukey show that k can reduce to 1.

Lacoss shows that the MLM has a comparable number of degrees of freedom to the Bartlett window spectrum, and Capon and Goodman (5) indicate, asymptotically, that MLM has the same level of stability Lacoss and Ulrych and Bishop (11) quote from a 1970 Stanford University PhD thesis by R Kromer, who proves that, in the asymptotic limit, MEM is:-

1) an unbiased estimate of the power spectrum

$$E(S_n(1)) = S(1)$$

2) a Gaussian distributed estimate

3) as a variance of $S_n(1)$

$$\text{Var}(S_n(1)) = 2/_k\, S^2(1)$$

where $k = N/_M$

Akaike also shows that $N/_M$ is the number of degrees of freedom, in his work on the autoregressive estimator. This is half the Bartlett and MLM values and seems low, since a small k means a larger range of fluctuation for the same fiducial limit. MEM, however, has a better resolution, and the trade-off between variability and resolution is not clear.

Examples of MEM Spectra

A routine is available on the Meteorological Office computer which calculates the Power Spectrum of a time series by MEM. It is stored in Met C 21 Load library in the module RSPECTRM.

The spectra given in this article are of synthetic time series calculated from sine functions and pseudo-random numbers to generate examples of line frequencies in Gaussian 'white' noise. The results show the variation of the spectra with order, length of data, frequency and noise content, and compares the MEM spectrum with the equivalent Fast Fourier Transform Spectrum.

Graphs 1 and 2 show variation with order of a number of different estimates of the power in a spectral peak. Estimates are based on the area under a curve (as Lacoss) and are obtained by

1) Fitting a Normal curve to the peak value and the half height (-3dB point) on the left and right of the peak separately.

2) Summing the area between both -3dB points by the Triangular rule and using this as the equivalent Gaussian value.

3) The mean of those three values.

4) An estimate obtained from the peak value as calculated by Lacoss.

Graph 1 uses a frequency step of 0.001 Hz in the Spectrum and the power estimates of these 5 values are shown for a 105 point section of a 0.71 Hz sine wave plus 10% amplitude of noise. The sampling rate is once every $\frac{1}{2}$ second.

At the 10th order, the curve is too wide and the estimates are too large. From orders 20 to 50 the Left and Right values have a wide variation, but the Integration is almost the same as the mean of the three values. This shows that although the peak may seem skewed, the area remains constant. The values from 60 to 100 show an initial rise and then a fall. The order is too high and the

noise peaks are boosted beyond their importance. The sharpening of the peak makes it fall between frequency points. The estimate from the peak value varies with the other estimates, but has a larger deviation, as expected.

The same data was analysed with a frequency step 1/10th of the first case. The results are displayed on Graph 2.

The value of the mean estimate is constant to $\pm$ 10% over the whole range above 20, and the differences between the left and right estimates are small until the order rises above 50, showing that the 'skewness' of Graph 1 was due to the large frequency step. The peak-derived value has a much greater variation than the other points. The best choice of order is between 30 and 40 and in this example, the frequency spacing of Graph 1 gives a good estimate.

The numerical experiments S1 to S10 have a data length of 64 points. The time series is of one or two sine waves and noise of varying amplitudes.

Experiments S1 and S2 involve the same time series with a 0.71 Hz wave and 1/10th amplitude noise. In S1, orders are chosen by the Akaike criterion and the 'third of the data length' rule. The MEM spectral peak is four orders of magnitude greater than the noise, and the frequency is accurate to 2%. The FFT has only 32 points, but is not correct even to that accuracy.

S2 shows MEM from orders 5 to 60. The FFT series S2a shows why a cut off at about half the record length is necessary to operate the criterion. A minimum is observed at order 11, but as the order is increased, a deeper minimum occurs at 56. From order 40 onwards, the noise peaks are progressively sharpened, until at 60 the peak has split into two and the noise amplitude is 1/20th of the peak value. Graph 3B shows the mean area estimate for S2. This has a large variance, which is due to the peak being finer than the frequency step, as is shown by 3A, which has a step length one tenth of B.

Experiments S3 and S4 have the same format as S1/2, but the noise amplitude is equal to the sine wave amplitude. Because of the high noise, the FPE criterion gives a very low value, but M = N/3 gives the frequency to a few per cent and a peak value five times the noise. The FFT does show power at the correct frequency, but the peak value is only 20% larger than the noise. Above order 40, the noise peaks are sharpened and are sometimes more than half the size of the signal peak.

S6 has a noise amplitude twice that of the signal. Neither MEM nor FFT resolve the signal. Only the FPE series and the autocorrelations are shown for S6, but the sharp drop in the FPE value for over large order is very noticeable.

Experiments S7 to 10 are grouped as S7/8 and S9/10. They are of time series composed of two sine functions of 0.69 Hz and 0.71 Hz. In the first group the noise amplitude is the same as each sine amplitude. Only the mean frequency is resolved but a subharmonic at 0.35 Hz has appeared. The FFT also has this effect, but has a poorer resolution. This phenomenon may be due to the harmonic beating being drowned by the particular noise pattern of this time series.

S9/10 shows the N/3 order resolving the two line frequencies from noise with one tenth the sine amplitude. The Akaike criterion does not resolve the pair and the FFT cannot resolve them because of the limitation on the number of points. The two lines are resolved by MEM from order 15 onwards, although after 40 the spectrum is obviously unstable.

Graphs 3 and 4 show the variation of the area and frequency estimates of the two peaks with spectral order. Although the peak values may be rather different, the area estimates are quite similar. The frequency estimates are consistently low, but by only a small amount. The choice of the FPE criterion will not be

critical since S10b shows that the FFT series has a wide valley and the minimum is a result of random variations.

Experiments 1.1 to 1.3 have spectra of a time series made from four sine functions and noise. 1.1 has 128 points, 1.2 has 64 and 1.3 has 64 points but the frequency difference between the two central peaks is smaller.

1.1 resolves all four lines on both MEM and FFT, although the low frequency on MEM has a second spike beside it. The peak values and area estimates are rather different for the four lines, but this is due to the frequency step swamping the sharpness of the peak.

In 1.2 the FFT cannot resolve the central peaks, while MEM still gives accurate estimates. When the separation is smaller, as in 1.3, even MEM cannot resolve the two lines.

Amplitude modulation and the production of sidebands is the subject of experiments 1.5 and 1.6. The first has the larger sideband separation. The linear MEM spectrum does not pick up the sidebands because of the large amplitude difference.

These experiments reinforce the work of other authors and shows the capabilities of the MIDAS program RSPECTRM. MEM is capable of a greater resolution than other methods, and the greater variance of the Power estimate is off-set by taking the area of a peak. The Akaike criterion is a useful method for choosing the spectral order, although for very periodic data, the choice is rather low.

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In the following results the time step was 0.5 units (secs) making the Nyquist Frequency = 1.0 Hz. The range of the frequency axis is 0.0→ 1.0 Hz linear.

S1 Sin wave 0.71 Hz Amplitude 1.0, Gaussian white noise $\sigma_z = 0.1$

a) Time series 64 points

b) Akaike FPE series (Logarithmic) up to order 2 )

c) Autocorrelations

d) Logarithmic spectrum FPE minimum 11th order

d) Linear spectrum 11th order

d) Linear spectrum 11th order

f) Log Spectrum N/3 21st order

g) Lin Spectrum 21st order

h) Fast Fourier Transform. Modulus.

S2 is the same data as S1 but the spectrum is sampled every 5th order

a) FPE series up to 60th order

b) Autocorrelation up to 60th order

c) to n) orders 5 (5) 60. Log and Lin spectra.

S3 Sin wave 0.71 Hz Amplitude 1.0. Noise $\sigma_z = 1.0$

a) 64 pt time series

b) FPE series to 21st order

c) Autocorrelations

d) Log spectrum at FPE minimum - order 3

e) Lin spectrum 3rd order

f) Log spectrum N/3 21st order

g) Lin spectrum 21st order

h) Fast Fourier Transform. Modulus.

S4 same data as S3 sampled every 5th order

a) FPE series to 60th order

b) Autocorrelations

c) to m) Log and Lin Spectra 10 (5) 60 orders.

S5 Sin wave 0.71 Hz amplitude 1.0 Noise $\sigma_z$ = 0.1

a) Time series 64 points

b) FPE series to 21st order. Notice that FPE min is 2

c) Autocorrelations 21st order

d) Log Spectrum. N/3 = 21st order

e) Lin Spectrum 21st order

f) Fast Fourier Transform. Modulus

S6 Same data

g) FPE series to 60th order

h) Autocorrelations to 60th order

S7 2 sin waves amplitudes 1.0 0.71 Hz and 0.69 Hz Noise $\sigma_z$ = 1.0

a) Time Series 64 pts

b) FPE Series to 21st order

c) Autocorrelations to 21st order

d) Log Spectrum FPE min 10th order

e) Lin Spectrum 10th order

f) Log Spectrum N/3 21st order

g) Lin Spectrum 21st order

h) Fast Fourier Transform

S8 Same series

i) FPE series to 60th order

S9 2 sin waves, amplitudes 1.0 0.71 Hz and 0.69 Hz Noise $\sigma_z$ = 0.1

a) Time series 64 pts

b) FPE series to 21st order

c) Autocorrelations

d) Logarithmic Spectrum. FPE min 12th order

e) Lin Spectrum 12th order

f) Log Spectrum N/3 21st order

g) Lin Spectrum N/3 21st order

h) Fast Fourier Transform.

S10 Same data as S9) Spectra samples every 5th order.

a) Time series as in S9)a)

b) FPE series to 60th order

c) Autocorrelations

d) to o) Log spectra orders 5(5)60.

1.1) 4 sin waves amplitudes 1.0 0.81Hz, 0.51 Hz, 0.49 Hz 0.19 Hz

Gaussian White Noise $\sigma^2$ = 0.1

a) Time series 128 pts

b) FPE series to 50 orders

c) Autocorrelations

d) Log spectrum. FPE min 40th order

e) Lin spectrum 40th order

f) Log spectrum N/3 42nd order

g) Lin spectrum 42nd order

h) Fast Fourier transform

1.2) Same data as 1.1 reduced to 64 points

a) Time series

b) FPE series

c) Autocorrelations

d) Log spectrum 21st order N/3

e) Lin spectrum 21st order

f) Log spectrum 24th order FPE min

g) Lin spectrum 24th order

h) Fast Fourier transform

1.3) 4 sin waves amplitudes 1.0 0.81 Hz, 0.505 Hz, 0.495 Hz, 0.19 Hz

Noise $\sigma^2$ = 0.1

a) Time series 64 points

b) FPE series to 30 orders

c) Autocorrelations

d) Log spectrum 21st order N/3

e) Lin spectrum 21st order

f) Log spectrum 24th order FPE minimum

g) Lin spectrum 24th order

h) Fast Fourier transform

1.5) Modulated sin wave. Calculated as $(1.0 + \cos(2\pi Bt))\cos(2\pi At) + N(0, \sigma_z^2)$ where A is 0.51 Hz and B is 0.49 Hz $\sigma_z = 0.1$

a) Time series 128 points

b) FPE series to 50 lags

c) Autocorrelations

d) Log spectrum FPE min 24 lags

e) Lin spectrum 24 orders

f) Log spectrum N/3 42 orders

g) Lin spectrum 42nd order

h) Fast Fourier transform

1.6) Modulated sin wave as in 1.5) with A = 0.5 Hz B = 0.11 Hz $\sigma_z = 0.1$

a) Time series 128 pts

b) FPE series to 50 orders

c) Autocorrelations

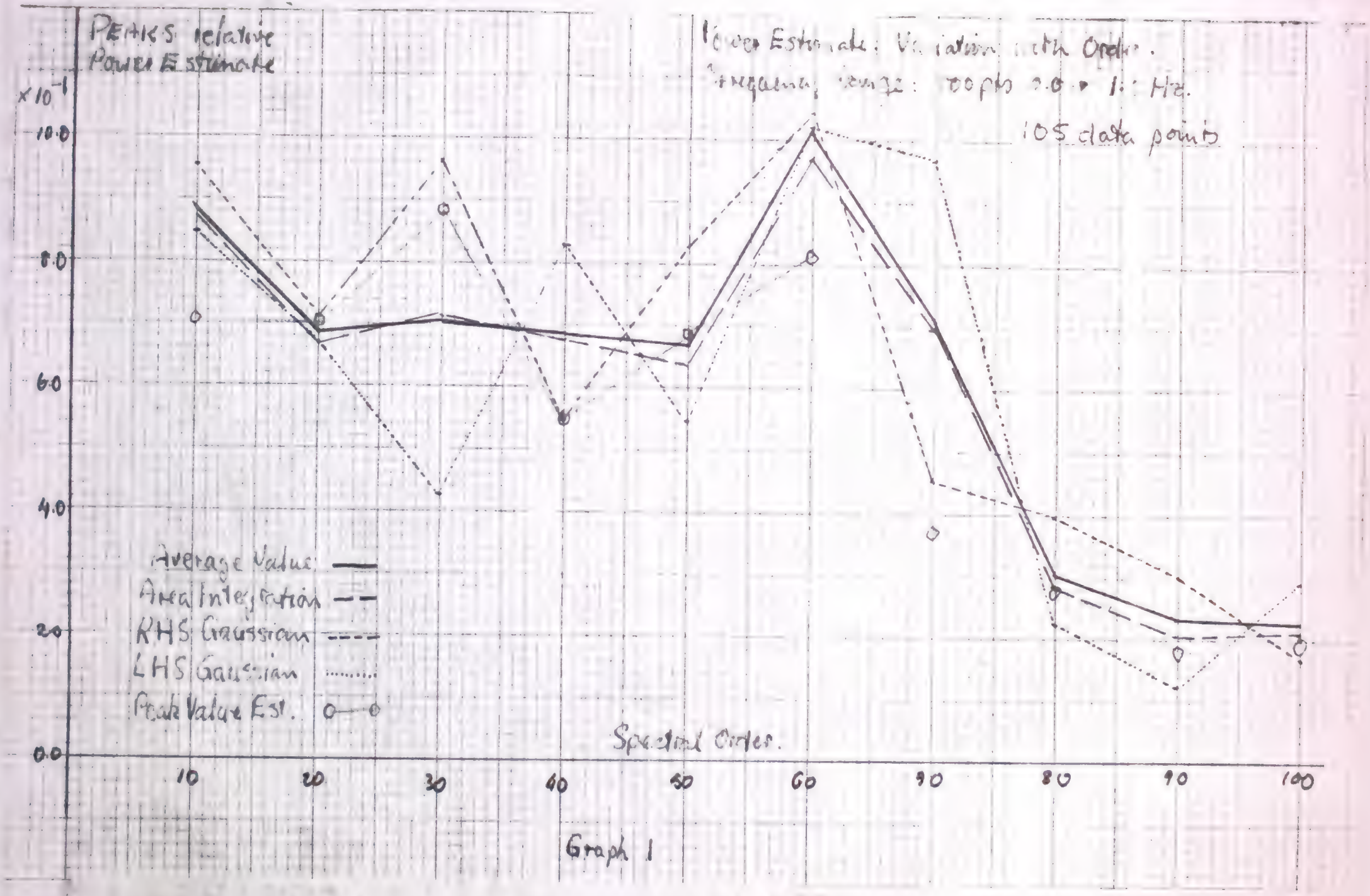
d) Log spectrum FPE min 24th order

e) Lin spectrum 24th order

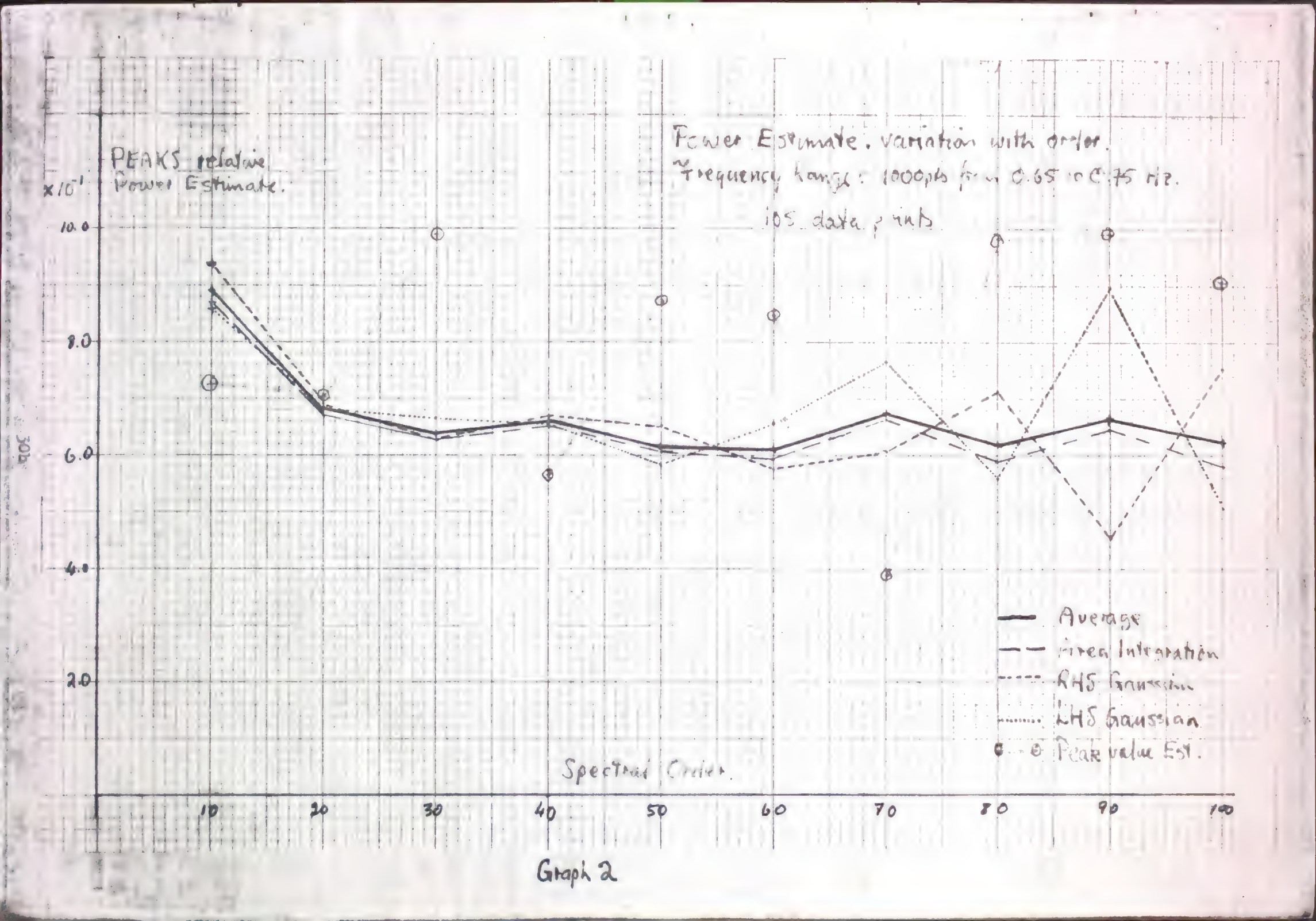
f) Log spectrum 42nd order N/3

g) Lin spectrum 42nd order

h) Fast Fourier transform

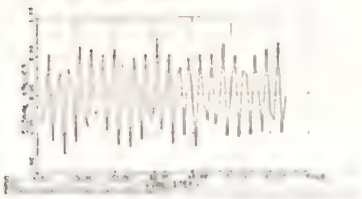

PEAKS relative
Power Estimate
×10⁻¹
10.0
8.0
6.0
4.0
2.0
0.0
Average Value
Area Integration
RHS Gaussian
LHS Gaussian
Peak Value Est.
Spectral Order.
10
20
30
40
50
60
70
80
90
100
105 data points

Graph 1


PEAKS relative
Power Estimate.
x10¹
10.0
8.0
6.0
4.0
2.0
Power Estimate. variation with order.
Frequency range: 1000pts from 0.65 to 0.75 Hz.
105 data points
Average
Area Integration
RHS Gaussian
LHS Gaussian
Peak value Est.
Spectral Order.
10
20
30
40
50
60
70
80
90
100

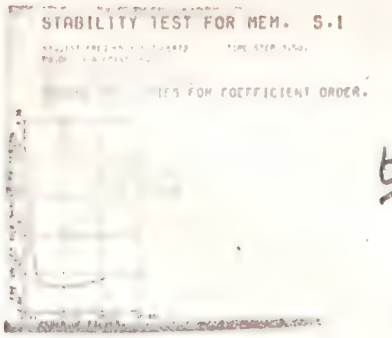
Graph 2

STABILITY TEST FOR MEM. S.1



a/

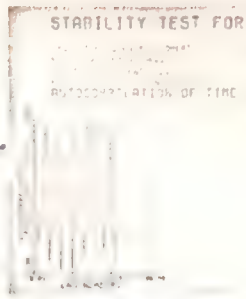
STABILITY TEST FOR MEM. S.1



b/

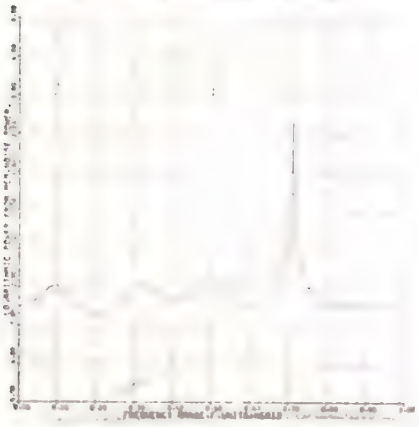
STABILITY TEST FOR MEM. S.1

AUTOCORRELATION OF TIME SERIES.



c/

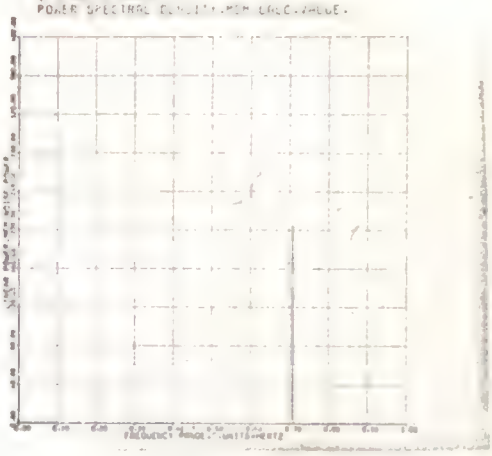
STABILITY TEST FOR MEM. S.1



d/

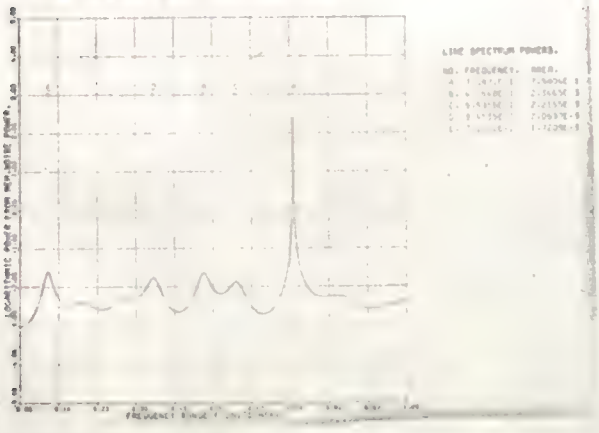
STABILITY TEST FOR MEM. S.1

POWER SPECTRAL DENSITY-MEM EXEC.VALUE.



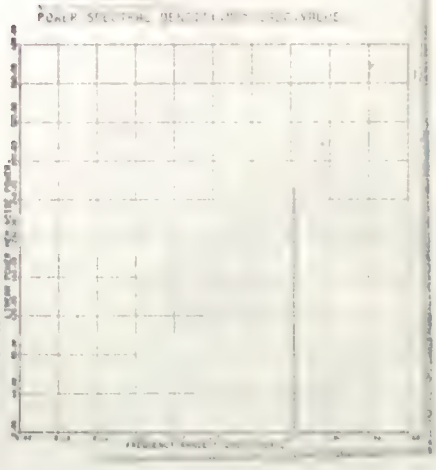
e/

STABILITY TEST FOR MEM. S.1



f/

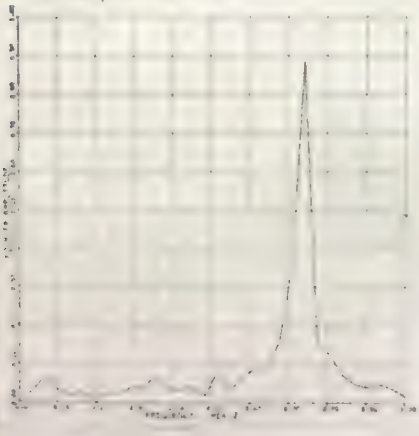
STABILITY TEST FOR MEM. S.1



g/

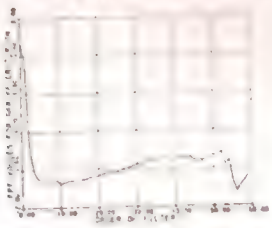
STABILITY TEST FOR MEM. S.1

FOURIER TRANSFORM OF DATA.



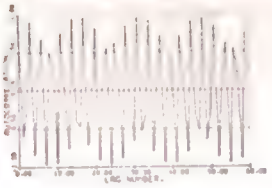
h/

STABILITY TEST FOR MEM. S.2



a)

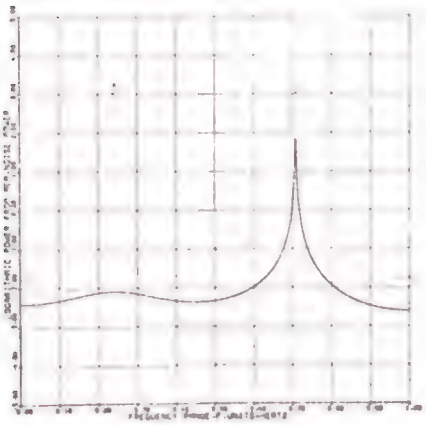
STABILITY TEST FOR MEM. S.2



b)

STABILITY TEST FOR MEM. S.2

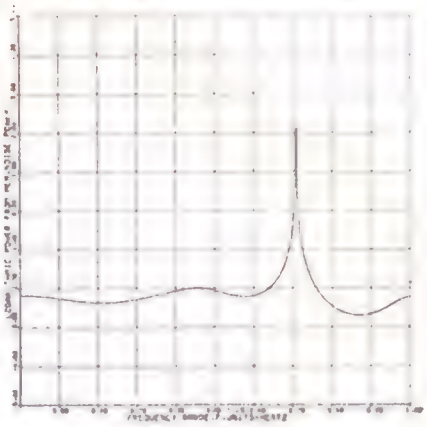
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



c)

STABILITY TEST FOR MEM. S.2

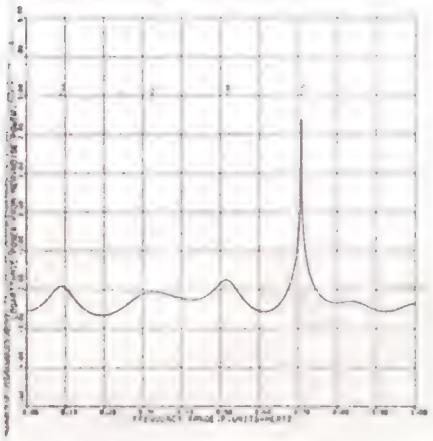
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



d)

STABILITY TEST FOR MEM. S.2

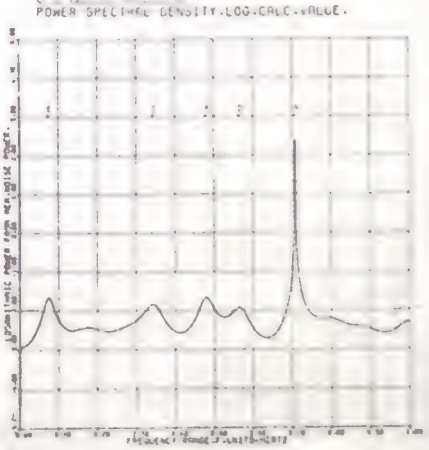
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



e)

STABILITY TEST FOR MEM. S.2

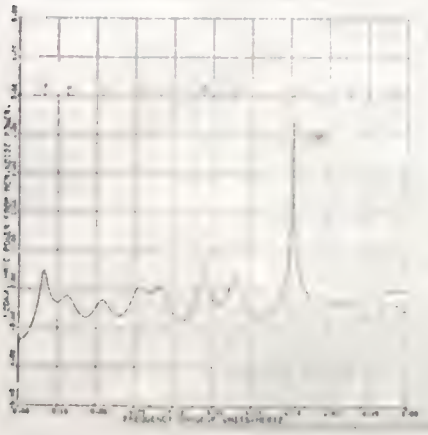
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



f)

STABILITY TEST FOR MEM. S.2

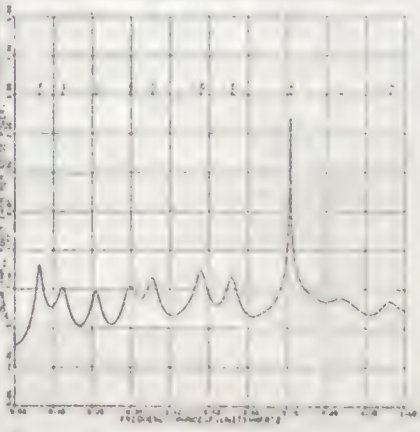
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



g)

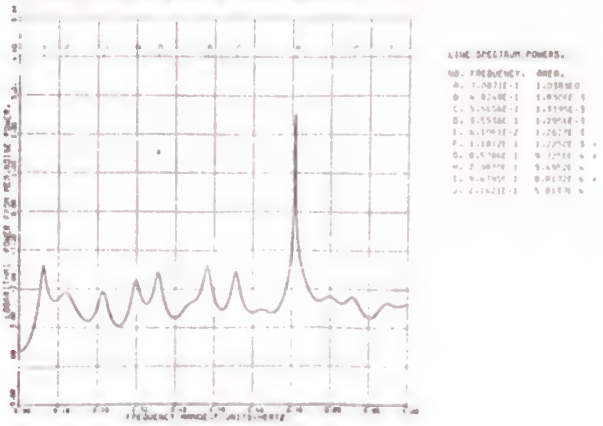
STABILITY TEST FOR MEM. S.2

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.

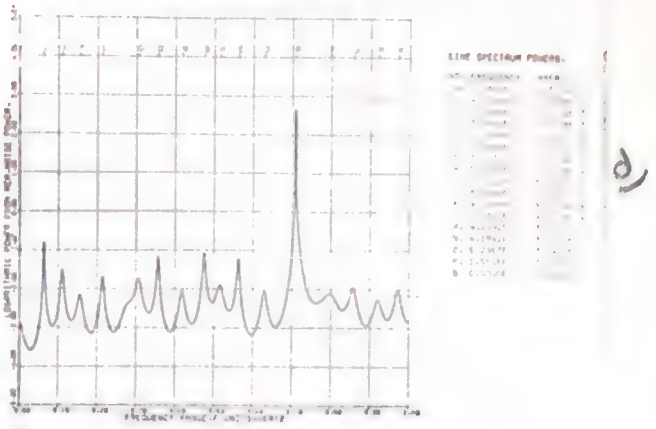


h)

STABILITY TEST FOR MEM. S.2

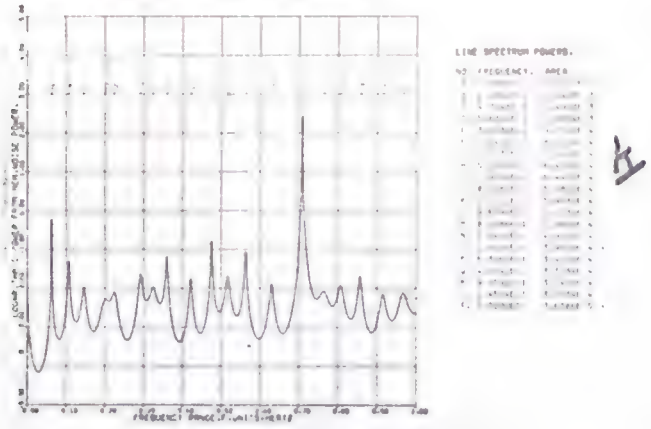


STABILITY TEST FOR MEM. S.2



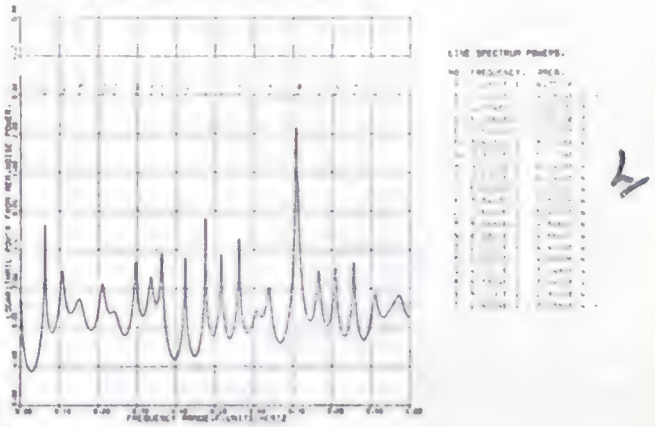
STABILITY TEST FOR MEM. S.2

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



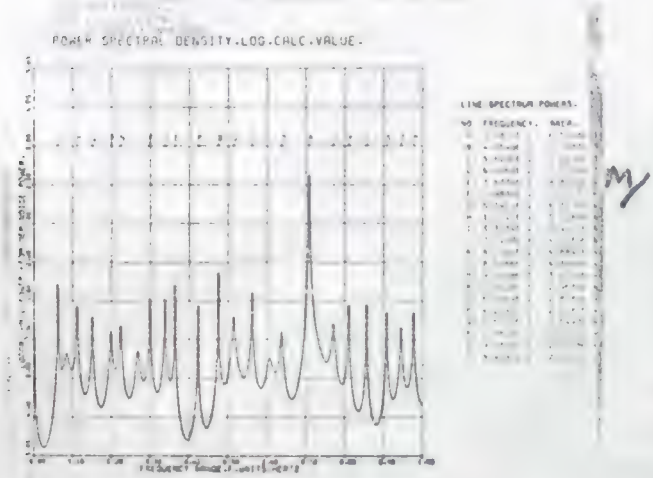
STABILITY TEST FOR MEM. S.2

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



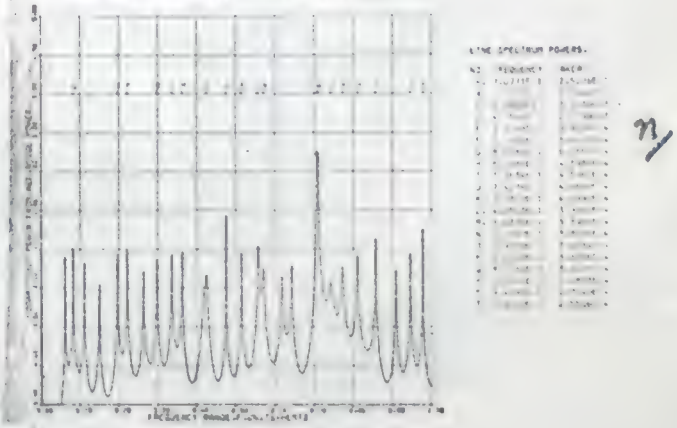
STABILITY TEST FOR MEM. S.2

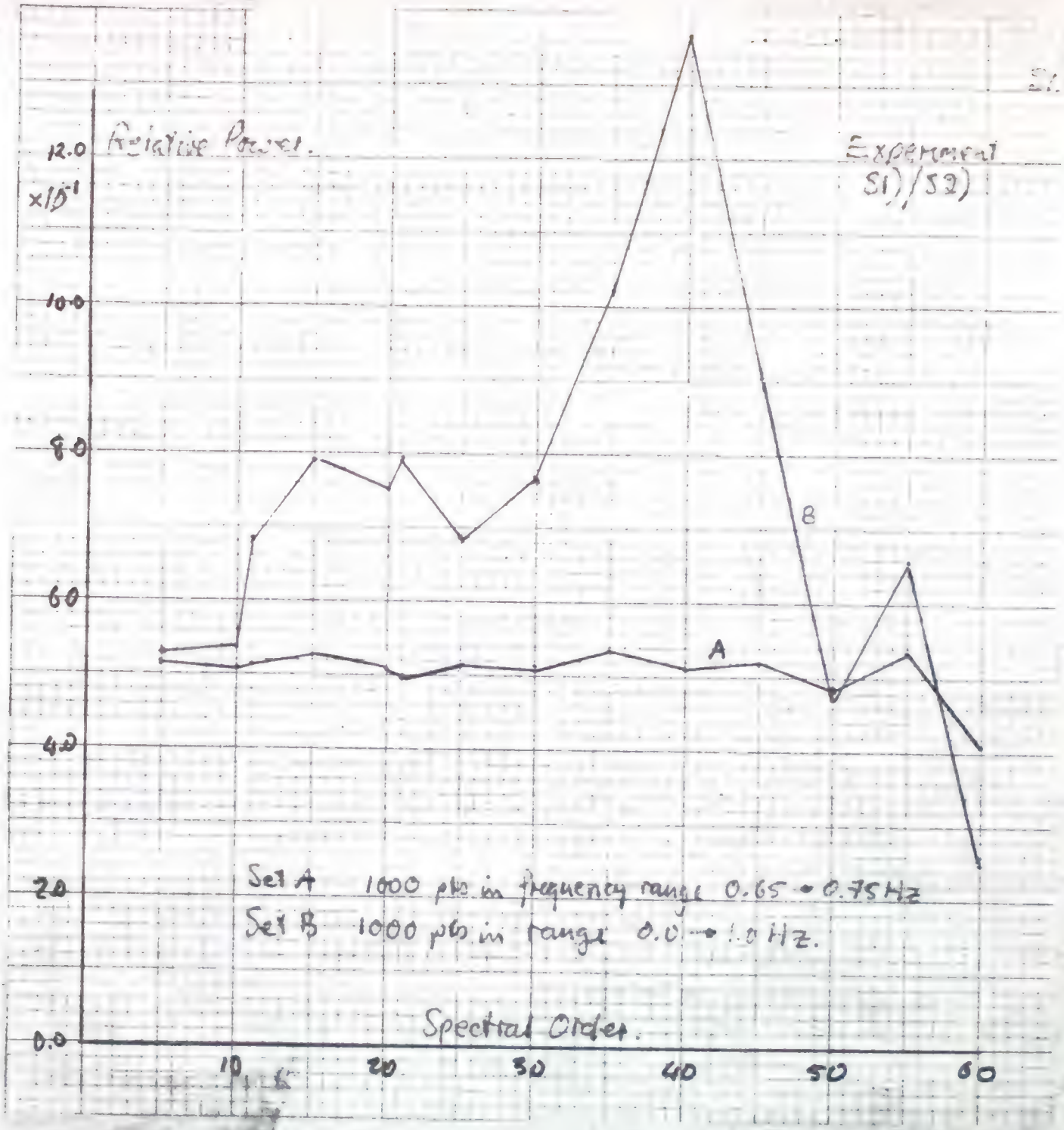
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



STABILITY TEST FOR MEM. S.2

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.




Relative Power.
×10⁻¹
12.0
10.0
8.0
6.0
4.0
2.0
0.0
Experiment
S1)/S2)
B
A
Set A 1000 pts in frequency range 0.65 → 0.75 Hz
Set B 1000 pts in range 0.0 → 1.0 Hz.
Spectral Order.
10
20
30
40
50
60

Graph 3

STABILITY TEST FOR MEM. S.3

TIME SERIES DATA.

a/

STABILITY TEST FOR MEM. S.3

AKAIKE FPE SERIES FOR COEFFICIENT ORDER.

b/

STABILITY TEST FOR MEM. S.3

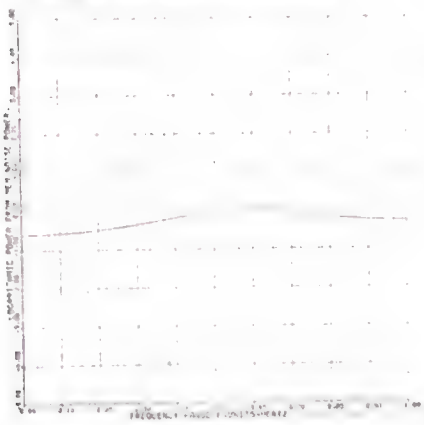
AUTOCORRELATION OF TIME SERIES.

c/

STABILITY TEST FOR MEM. S.3

POWER SPECTRAL DENSITY.

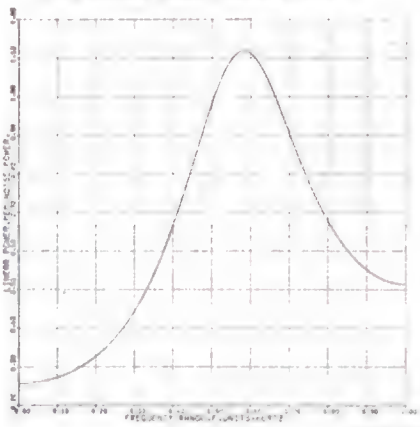
LINE SPECTRUM POWERS.



d/

STABILITY TEST FOR MEM. S.3

POWER SPECTRAL DENSITY.MEM.COEF.VALUE.

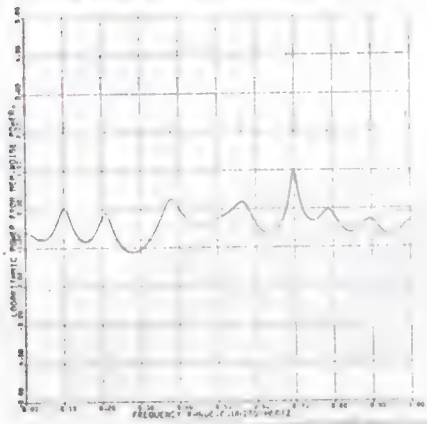


e/

STABILITY TEST FOR MEM. S.3

POWER SPECTRAL DENSITY.LOD-COEF.VALUE.

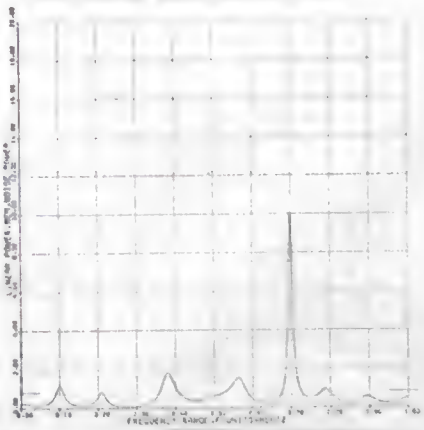
LINE SPECTRUM POWERS.



f/

STABILITY TEST FOR MEM. S.3

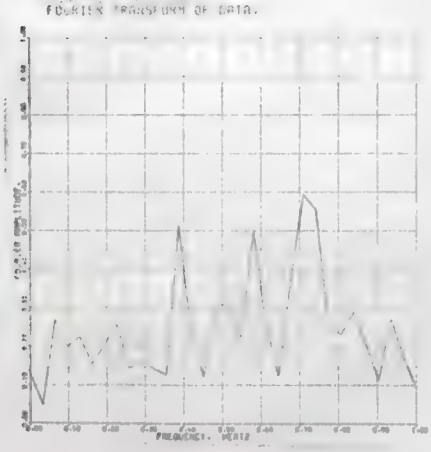
POWER SPECTRAL DENSITY.MEM.COEF.VALUE.



g/

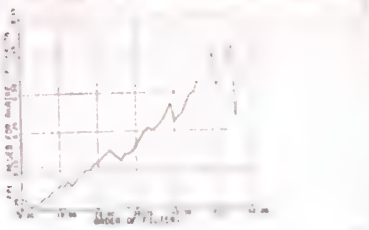
STABILITY TEST FOR MEM. S.3

FOURIER TRANSFORM OF DATA.



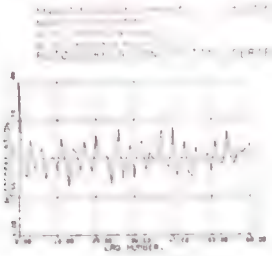
h/

STABILITY TEST FOR MEM. S.4



a/

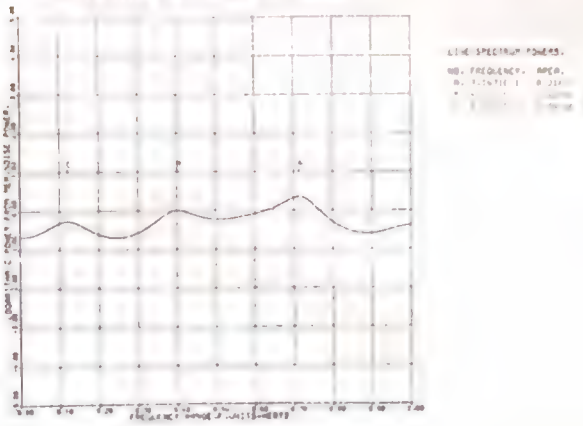
STABILITY TEST FOR MEM. S.4



b/

STABILITY TEST FOR MEM. S.4

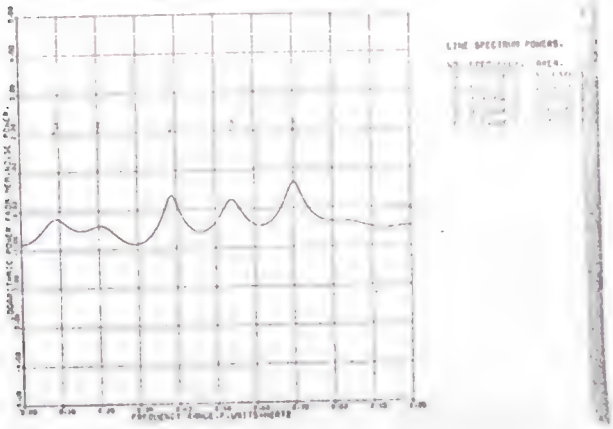
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



c/

STABILITY TEST FOR MEM. S.4

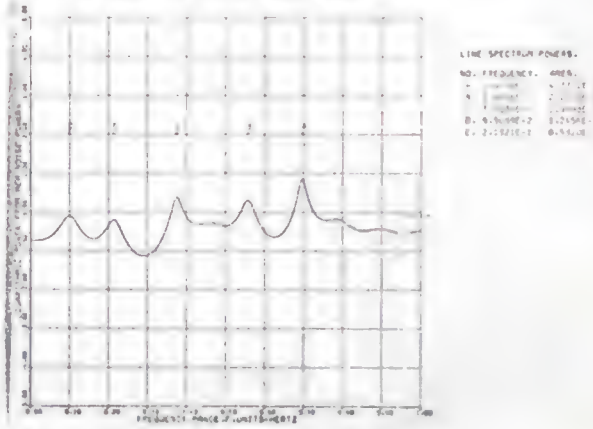
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



d/

STABILITY TEST FOR MEM. S.4

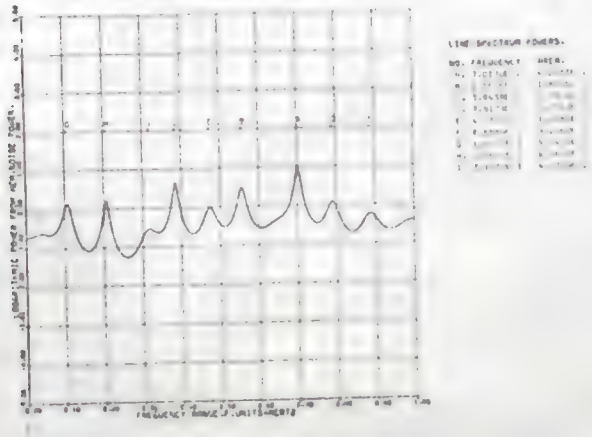
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



e/

STABILITY TEST FOR MEM. S.4

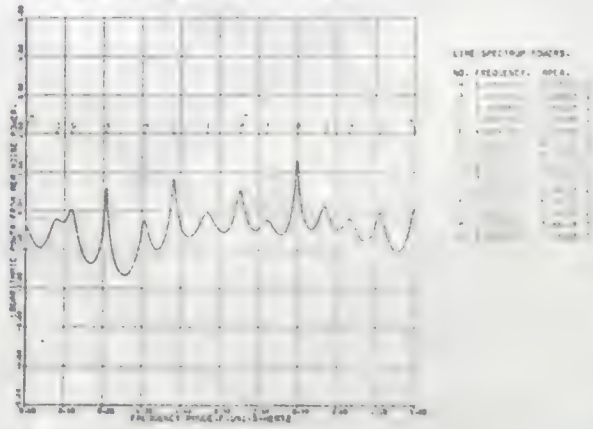
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



f/

STABILITY TEST FOR MEM. S.4

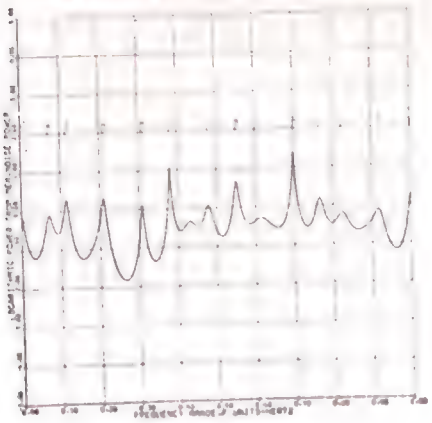
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



g/

STABILITY TEST FOR MEM. S.4

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.

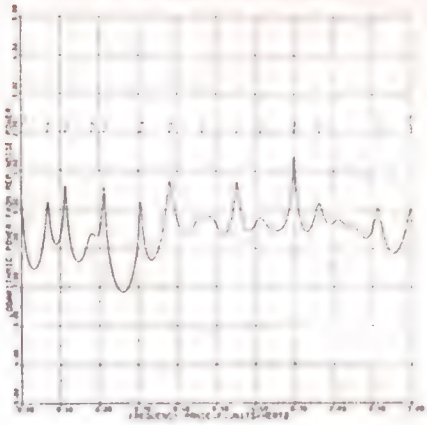


LINE SPECTRUM POWERS.

h

STABILITY TEST FOR MEM. S.4

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.

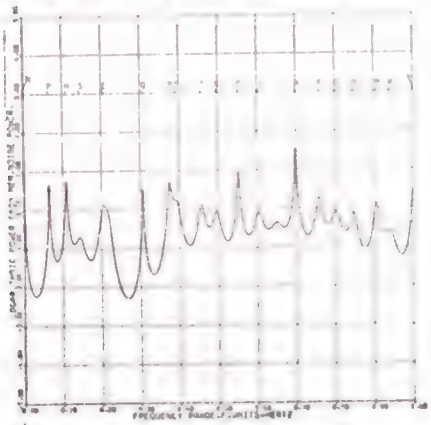


LINE SPECTRUM POWERS.

i

STABILITY TEST FOR MEM. S.4

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.

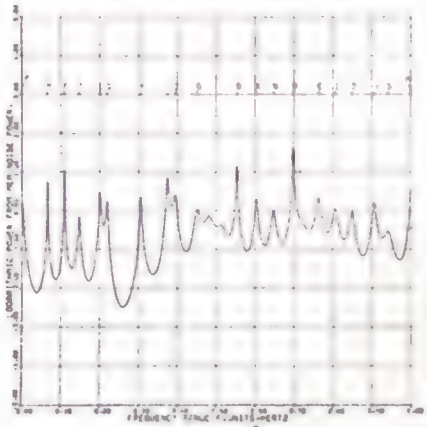


LINE SPECTRUM POWERS.

j

STABILITY TEST FOR MEM. S.4

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.

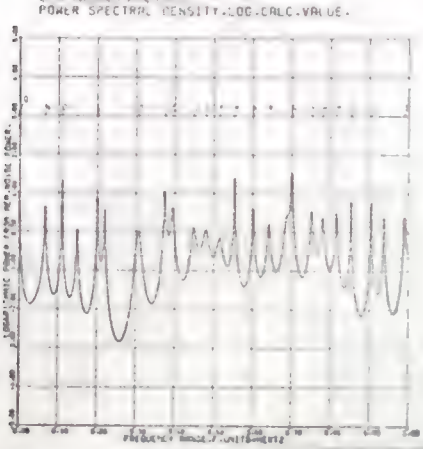


LINE SPECTRUM POWERS.

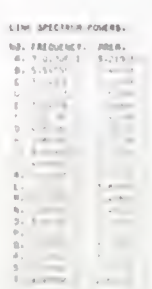
k

STABILITY TEST FOR MEM. S.4

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



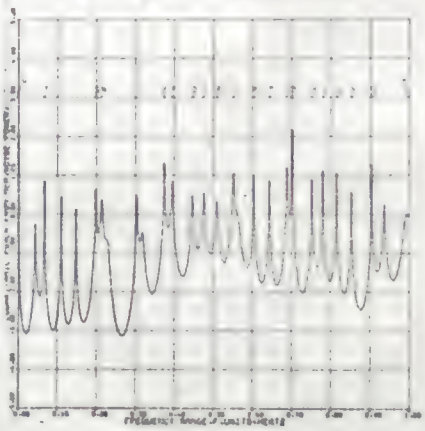
LINE SPECTRUM POWERS.



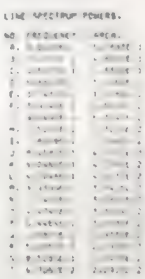
l

STABILITY TEST FOR MEM. S.4

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.

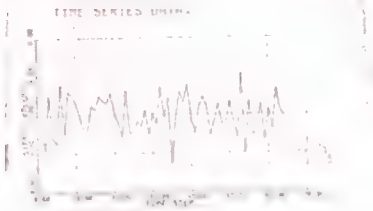


LINE SPECTRUM POWERS.



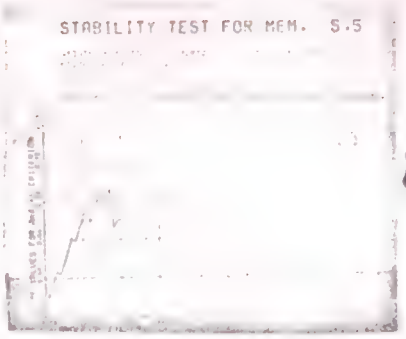
m

STABILITY TEST FOR MEM. S.5



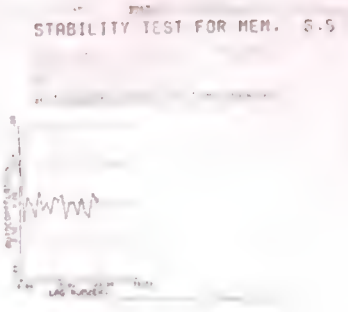
a

STABILITY TEST FOR MEM. S.5



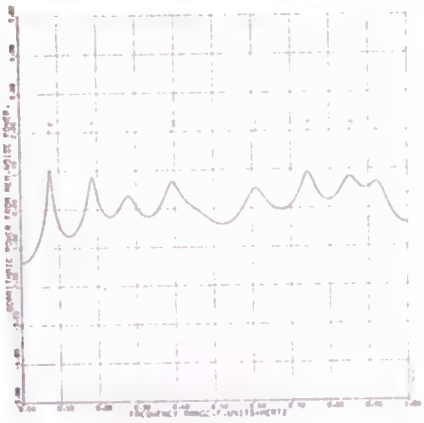
b

STABILITY TEST FOR MEM. S.5



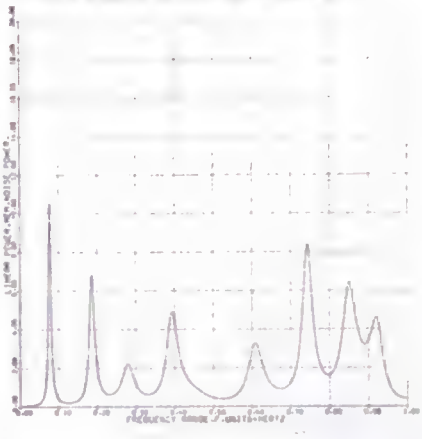
c

STABILITY TEST FOR MEM. S.5



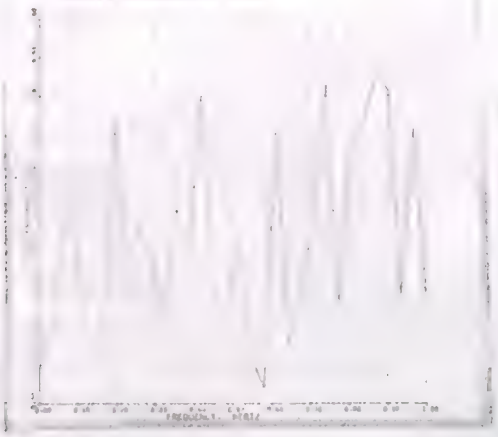
d

STABILITY TEST FOR MEM. S.5



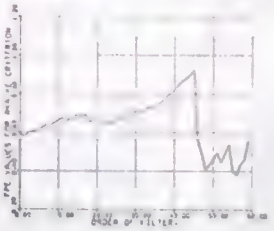
e

STABILITY TEST FOR MEM. S.5



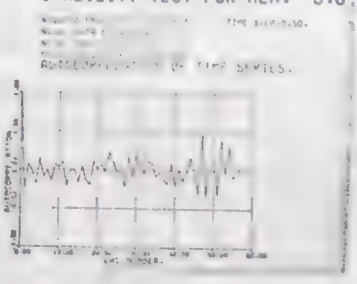
f

STABILITY TEST FOR MEM. S.6



g

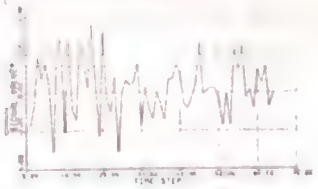
STABILITY TEST FOR MEM. S.6



h

STABILITY TEST FOR MEM. S.7

TIME SERIES DATA.



a/

STABILITY TEST FOR MEM. S.7

AKAIKE FPE SERIES FOR COEFFICIENT ORDER.

b/

STABILITY TEST FOR MEM. S.7

AUTOCORRELATION OF TIME SERIES.

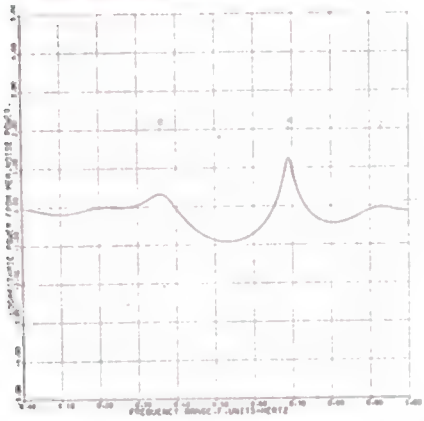
c/

STABILITY TEST FOR MEM. S.7

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.

LINE SPECTRUM POWERS.

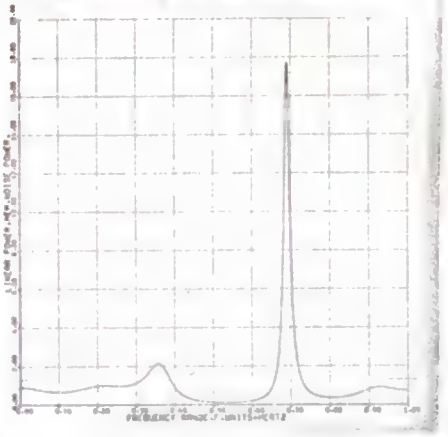
NO. FREQUENCY. AMPL.



d/

STABILITY TEST FOR MEM. S.7

POWER SPECTRAL DENSITY.MEM.CALC.VALUE.



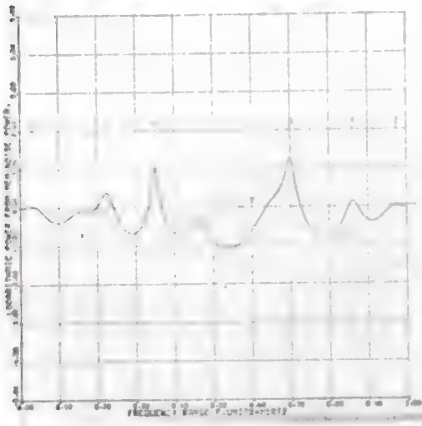
e/

STABILITY TEST FOR MEM. S.7

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.

LINE SPECTRUM POWERS.

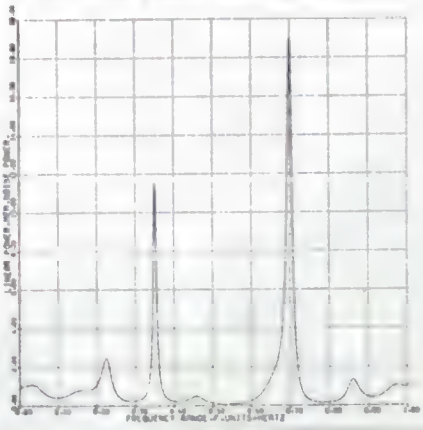
NO. FREQUENCY. AMPL.



f/

STABILITY TEST FOR MEM. S.7

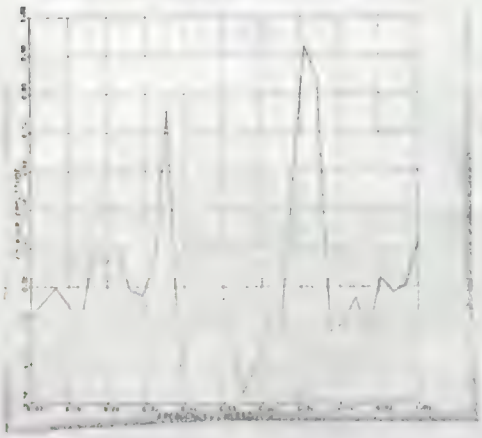
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.



g/

STABILITY TEST FOR MEM. S.7

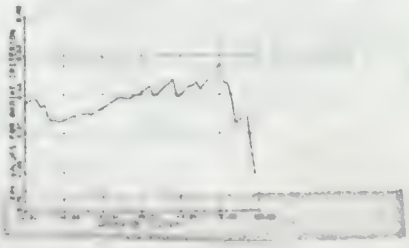
FOURIER TRANSFORM OF DATA.



h/

STABILITY TEST FOR MEM. S.8

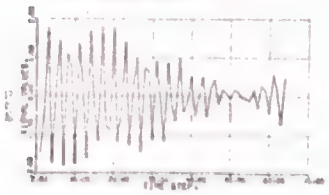
AKAIKE FPE SERIES FOR COEFFICIENT ORDER.



i/

STABILITY TEST FOR MEM. S.9

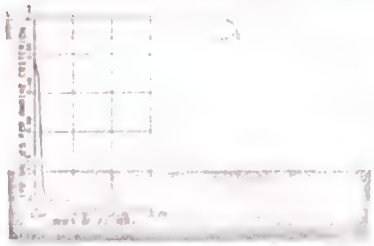
TIME SERIES DATA.



a/

STABILITY TEST FOR MEM. S.9

ARAIKE FPE SERIES FOR COEFFICIENT ORDER.



b/

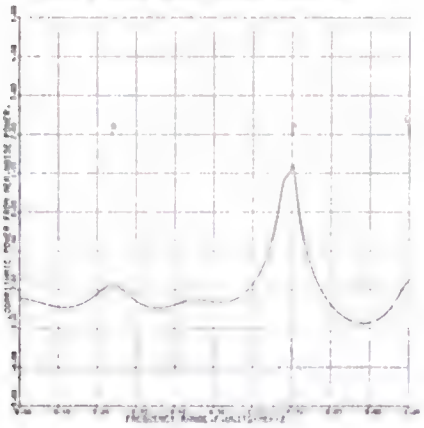
STABILITY TEST FOR MEM. S.9

AUTOCORRELATION OF TIME SERIES.

c/

STABILITY TEST FOR MEM. S.9

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



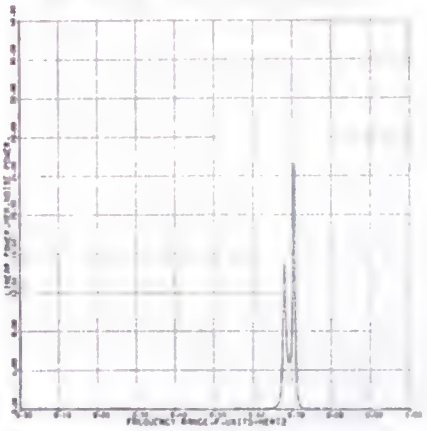
LINE SPECTRUM POWERS.

| NO. | FREQUENCY. | AMPL. |
|---|---|---|
| A. | 7.0175E-2 | 1.0558E-3 |
| B. | 2.1930E-1 | 3.8314E-3 |
| C. | 1.5000E0 | 2.2620E-3 * |

d/

STABILITY TEST FOR MEM. S.9

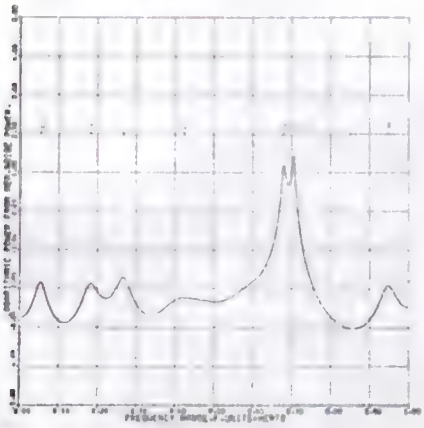
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.



e/

STABILITY TEST FOR MEM. S.9

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



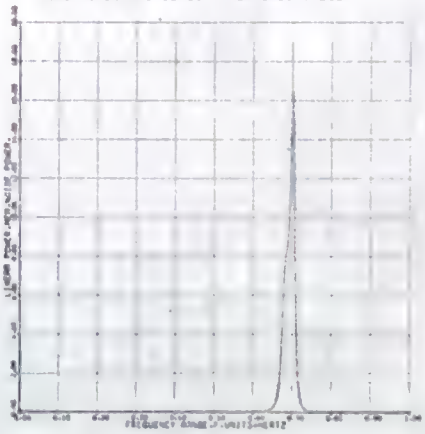
LINE SPECTRUM POWERS.

| NO. | FREQUENCY. | AMPL. |
|---|---|---|
| A. | 7.0175E-2 | 9.8504E-4 |
| B. | 6.0351E-1 | 6.6.9[illegible]E-3 |
| C. | 9.[illegible]E-1 | 1.[illegible]E-3 * |
| D. | 1.0[illegible]E-1 | 1.6[illegible]E-3 |
| E. | 1.0[illegible]E-1 | 1.6[illegible]E-3 |
| F. | 9.[illegible]E-1 | 1.6[illegible]E-3 |
| G. | 9.[illegible]E-2 | 1.[illegible]E-3 |

f/

STABILITY TEST FOR MEM. S.9

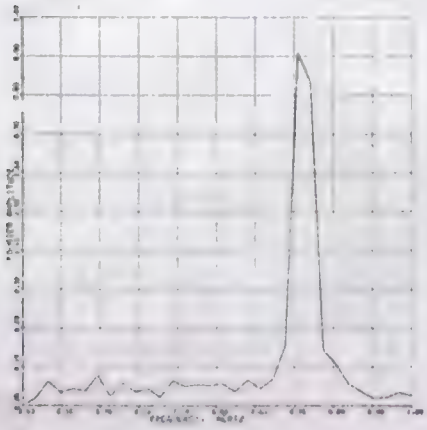
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.



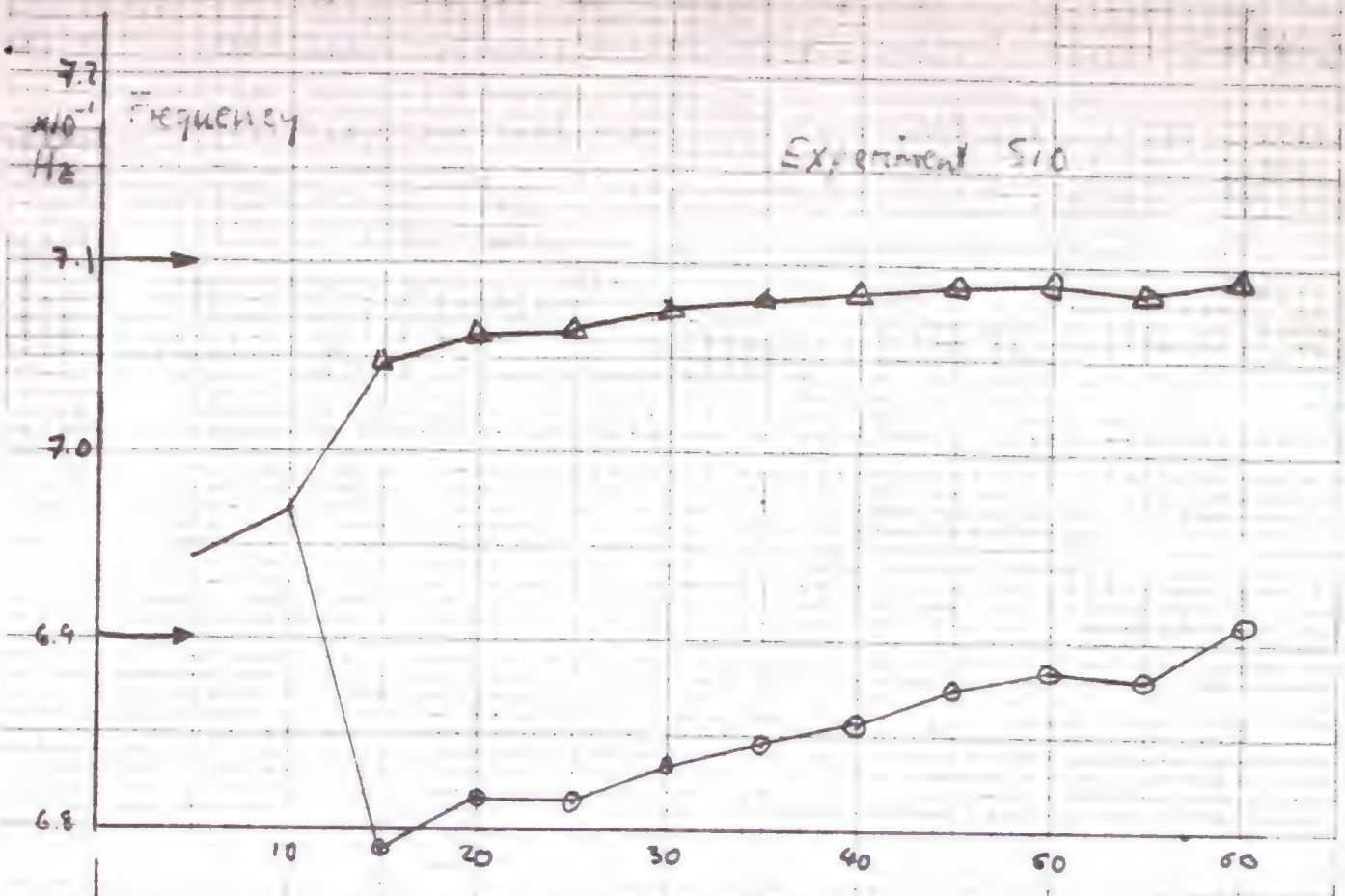
g/

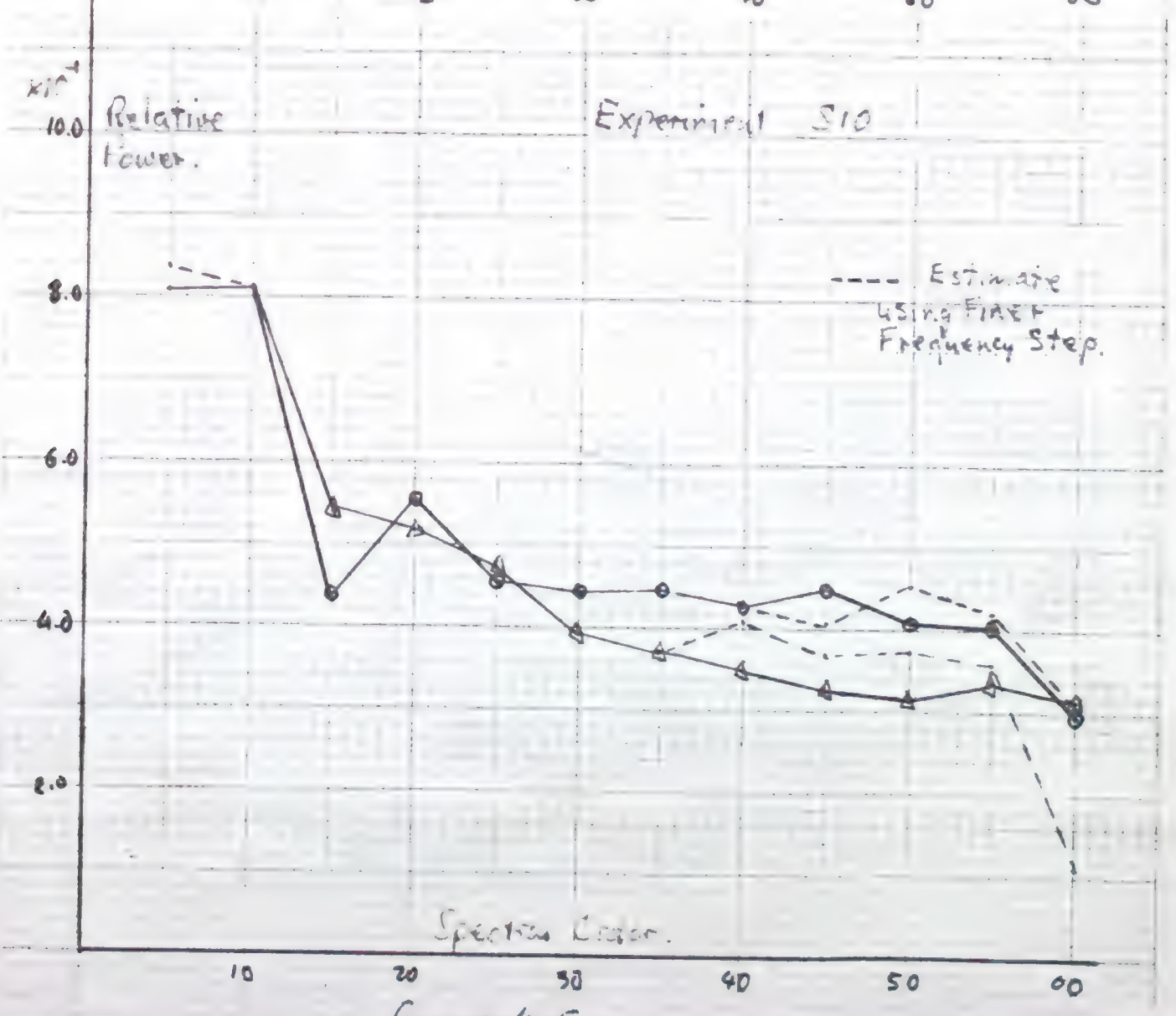
STABILITY TEST FOR MEM. S.9

FOURIER TRANSFORM OF DATA.



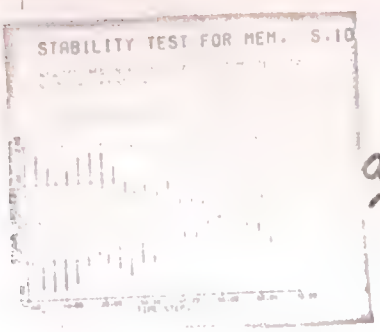
h/


7.2
x10⁻¹
Hz
Frequency
Experiment S10
7.1
7.0
6.9
6.8
10
20
30
40
50
60


x10⁻¹
10.0
Relative
Power.
Experiment S10
---- Estimate
using First
Frequency Step.
8.0
6.0
4.0
2.0
Spectrum Order.
10
20
30
40
50
60

Graphs 4, 5.

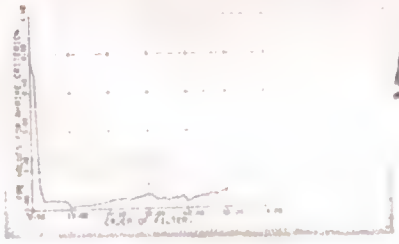
STABILITY TEST FOR MEM. S-10



a/

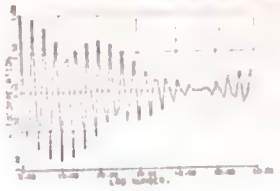
STABILITY TEST FOR MEM. S-10

AKAIKE FPE SERIES FOR COEFFICIENT ORDER.



b/

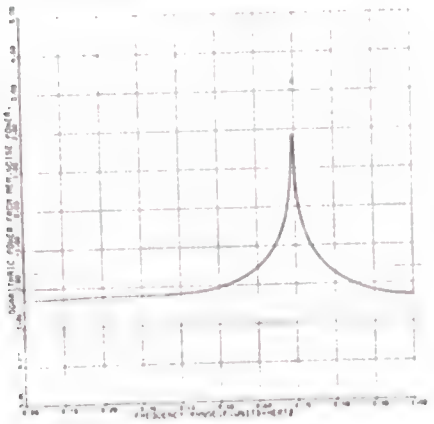
STABILITY TEST FOR MEM. S-10



c/

STABILITY TEST FOR MEM. S-10

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



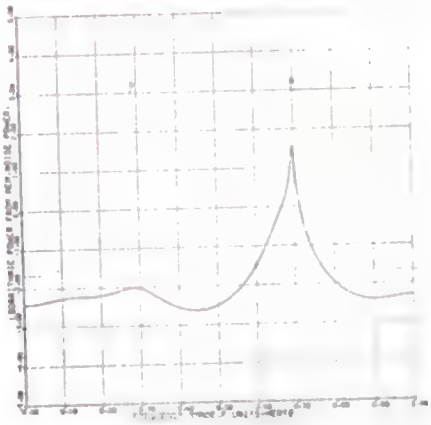
LINE SPECTRUM POWERS.

NO. FREQUENCY. AREA.

d/

STABILITY TEST FOR MEM. S-10

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.

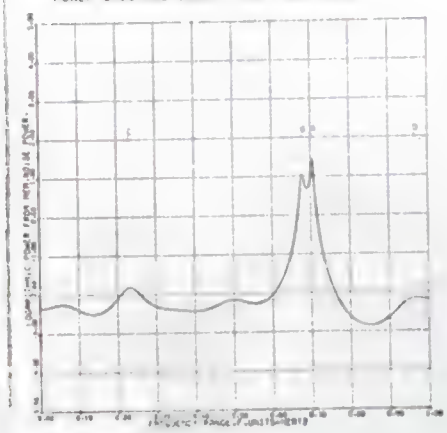


LINE SPECTRUM POWERS.

e/

STABILITY TEST FOR MEM. S-10

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



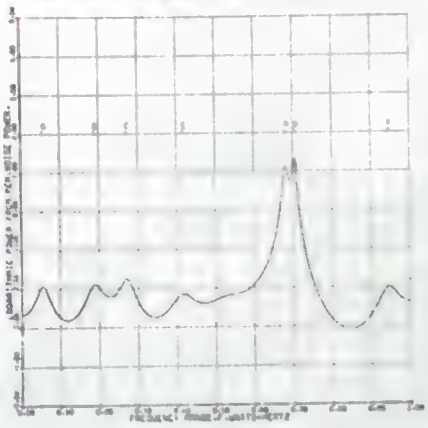
LINE SPECTRUM POWERS.

NO. FREQUENCY. AREA.

f/

STABILITY TEST FOR MEM. S-10

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.

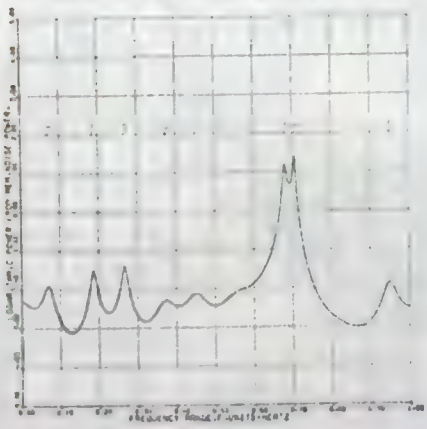


LINE SPECTRUM POWERS.

g/

STABILITY TEST FOR MEM. S-10

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.

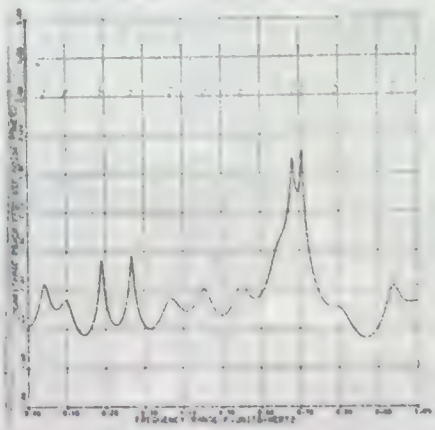


LINE SPECTRUM POWERS.

h/

STABILITY TEST FOR MEM. S-10

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.

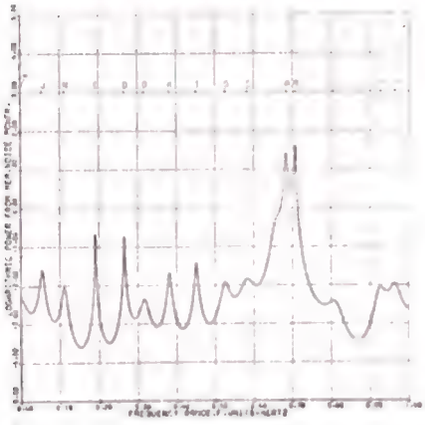


LINE SPECTRUM POWERS.

i/

STABILITY TEST FOR MEM. S.10

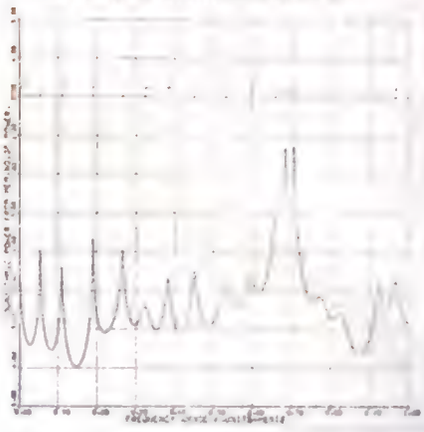
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



j

STABILITY TEST FOR MEM. S.10

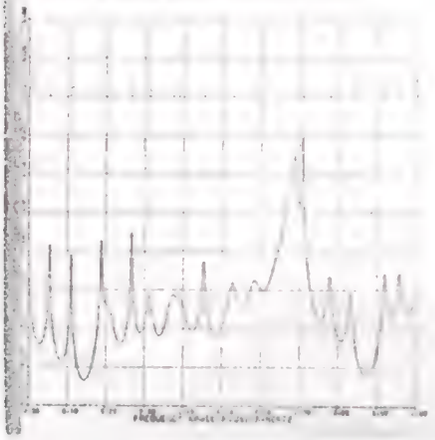
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



k

STABILITY TEST FOR MEM. S.10

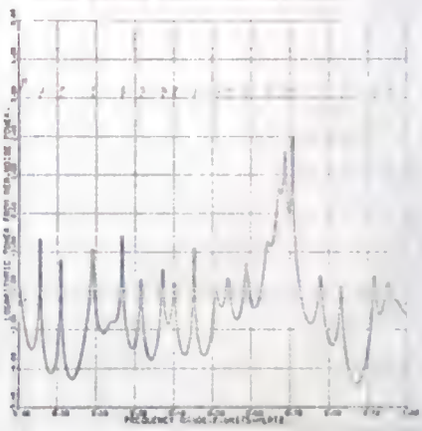
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



l

STABILITY TEST FOR MEM. S.10

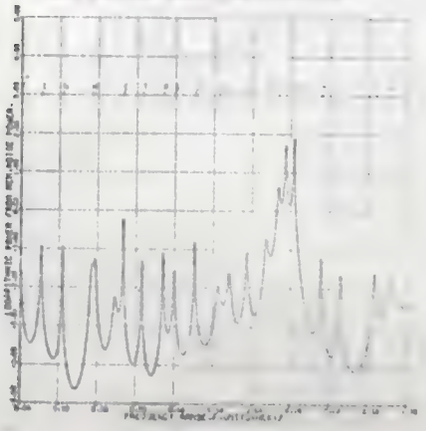
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



m

STABILITY TEST FOR MEM. S.10

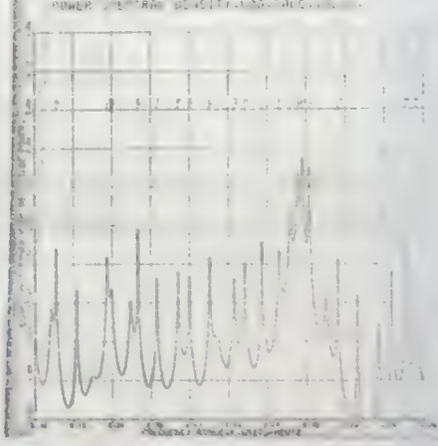
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



n

STABILITY TEST FOR MEM. S.10

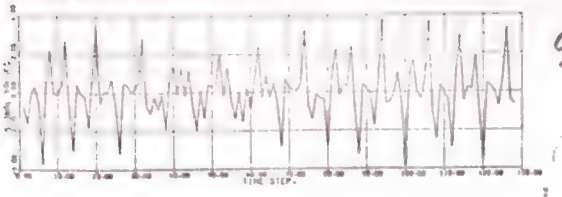
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



o

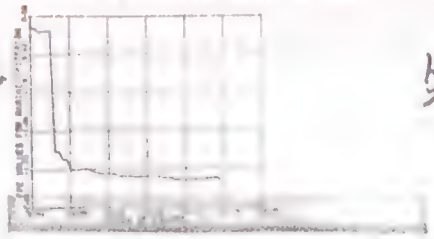
MEM. NUMERICAL EXPERIMENT. 1.1

TIME SERIES DATA.



a/

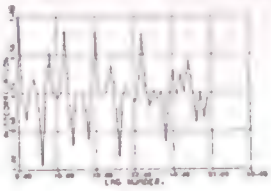
MEM. NUMERICAL EXPERIMENT. 1.1



b/

MEM. NUMERICAL EXPERIMENT. 1.1

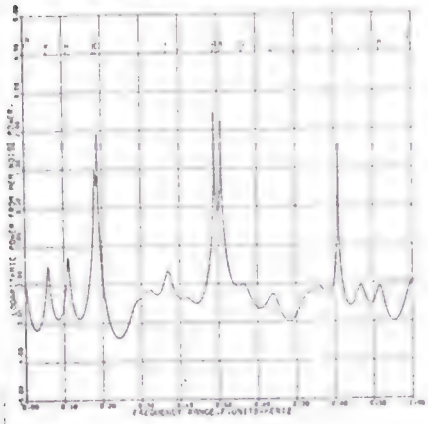
AUTOCORRELATION OF TIME SERIES.



c/

MEM. NUMERICAL EXPERIMENT. 1.1

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



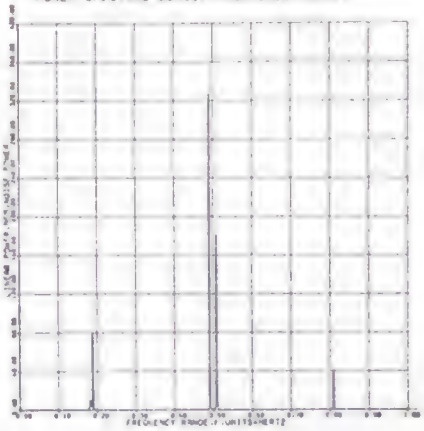
LINE SPECTRUM POWERS.

| NO. | FREQUENCY. | AREA. |
|---|---|---|
| 1 | | |
| 2 | | |
| 3 | | |
| 4 | | |
| 5 | | |
| 6 | | |
| 7 | | |
| 8 | | |
| 9 | | |
| 10 | | |

d/

MEM. NUMERICAL EXPERIMENT. 1.1

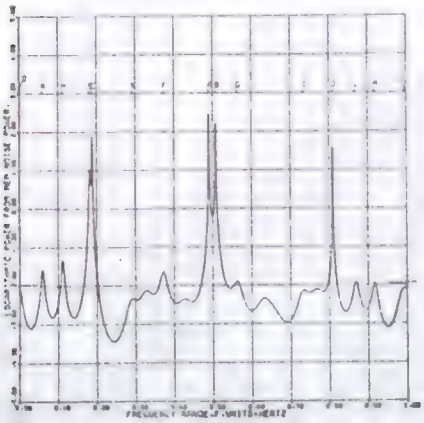
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.



e/

MEM. NUMERICAL EXPERIMENT. 1.1

POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



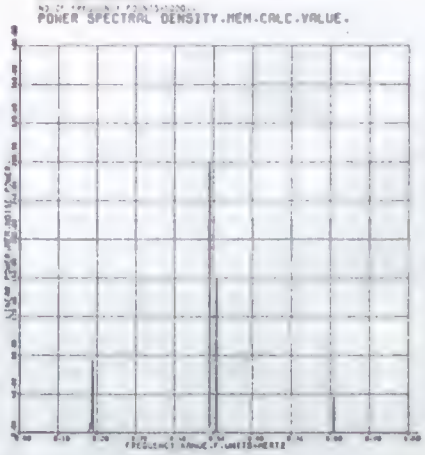
LINE SPECTRUM POWERS.

| NO. | FREQUENCY. | AREA. |
|---|---|---|
| A | | |
| B | | |
| C | | |
| D | | |
| E | | |
| F | | |
| G | | |
| H | | |
| I | | |
| J | | |
| K | 5.9059E-2 | 6.1060E-6 |
| L | 8.6897E-1 | 5.0592E-6 |
| M | 9.1792E-1 | 6.7880E-6 |
| N | | |
| O | | |

f/

MEM. NUMERICAL EXPERIMENT. 1.1

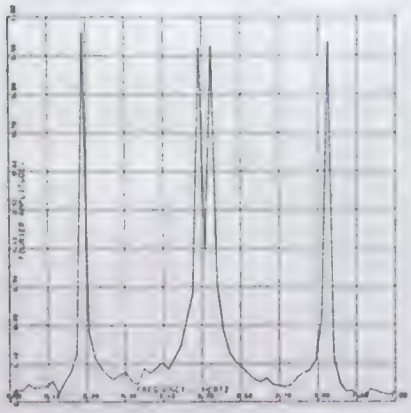
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.



g/

MEM. NUMERICAL EXPERIMENT. 1.1

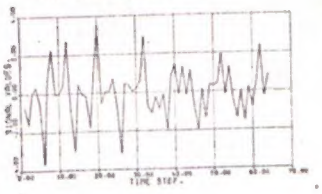
FOURIER TRANSFORM OF DATA.



h/

MEM. NUMERICAL EXPERIMENT. 1.2

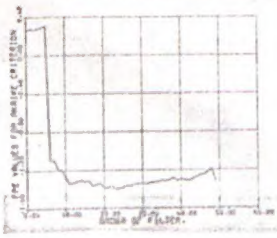
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50.
NO.OF DATA POINTS=64..
NO.OF COEFFICIENTS=50.
NO.OF FREQUENCY POINTS=1000..
TIME SERIES DATA.




a

MEM. NUMERICAL EXPERIMENT. 1.2

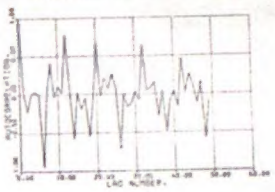
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50.
NO.OF DATA POINTS=64..
NO.OF COEFFICIENTS=50.
NO.OF FREQUENCY POINTS=1000..
AKAIKE FPE SERIES FOR COEFFICIENT ORDER.




b

MEM. NUMERICAL EXPERIMENT. 1.2

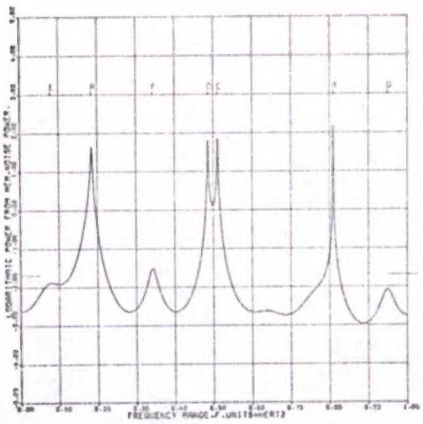
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50.
NO.OF DATA POINTS=64..
NO.OF COEFFICIENTS=50.
NO.OF FREQUENCY POINTS=1000..
AUTOCORRELATION OF TIME SERIES.




c

MEM. NUMERICAL EXPERIMENT. 1.2

NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=64..
NO.OF COEFFICIENTS=24.
NO.OF FREQUENCY POINTS=1000..
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



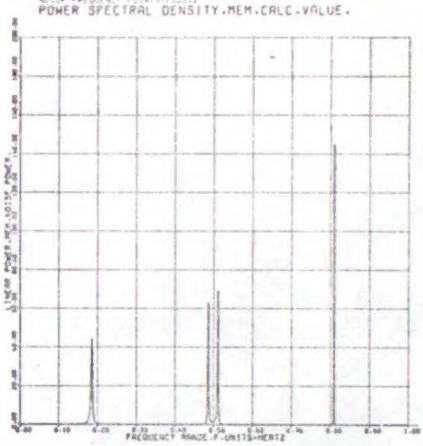

LINE SPECTRUM POWERS.

| NO. | FREQUENCY. | AREA. |
|---|---|---|
| A. | 1.8519E-1 | 5.4204E-1 |
| B. | 8.0981E-1 | 4.6507E-1 |
| C. | 5.1151E-1 | 4.3809E-1 |
| D. | 4.8649E-1 | 3.6276E-1 |
| E. | 7.9079E-2 | 2.6050E-1 |
| F. | 3.4334E-1 | 2.0762E-3 |
| G. | 9.4995E-1 | 8.5609E-4 |

d

MEM. NUMERICAL EXPERIMENT. 1.2

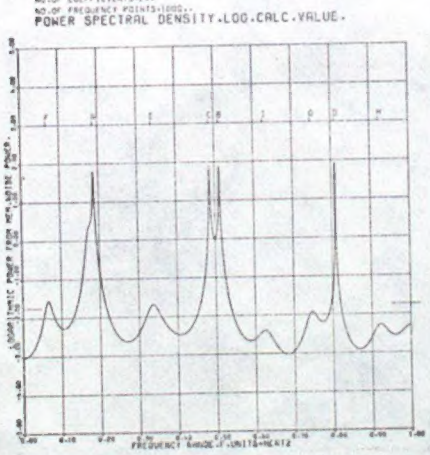
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=64..
NO.OF COEFFICIENTS=24.
NO.OF FREQUENCY POINTS=1000..
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.




e

MEM. NUMERICAL EXPERIMENT. 1.2

NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=64..
NO.OF COEFFICIENTS=24.
NO.OF FREQUENCY POINTS=1000..
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



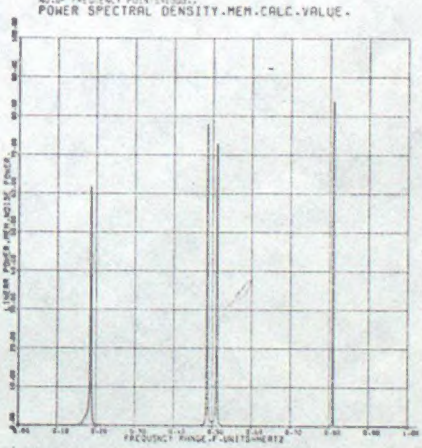

LINE SPECTRUM POWERS.

| NO. | FREQUENCY. | AREA. |
|---|---|---|
| A. | 1.8519E-1 | 4.8961E-1 |
| B. | 5.1151E-1 | 4.6514E-1 |
| C. | 4.8649E-1 | 3.8311E-1 |
| D. | 8.0981E-1 | 2.6547E-1 |
| E. | 3.3734E-1 | 2.4253E-3 |
| F. | 6.6066E-2 | 1.8053E-3 |
| G. | 7.4775E-1 | 1.5217E-3 |
| H. | 9.2292E-1 | 8.5694E-4 |
| I. | 6.2563E-1 | 7.0335E-4 |

f

MEM. NUMERICAL EXPERIMENT. 1.2

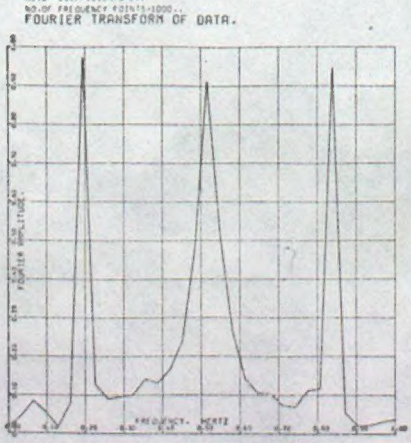
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=64..
NO.OF COEFFICIENTS=24.
NO.OF FREQUENCY POINTS=1000..
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.




g

MEM. NUMERICAL EXPERIMENT. 1.2

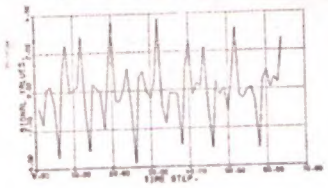
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=64..
NO.OF COEFFICIENTS=24.
NO.OF FREQUENCY POINTS=1000..
FOURIER TRANSFORM OF DATA.




h

MEM. NUMERICAL EXPERIMENT. 1.3

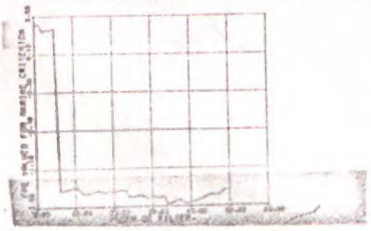
NYQUIST FREQUENCY=1.0HERTZ TIME STEP=0.50.
NO.OF DATA POINTS=64...
NO.OF COEFFICIENTS=50.
NO.OF FREQUENCY POINTS=1000...
TIME SERIES DATA.




a/

MEM. NUMERICAL EXPERIMENT. 1.3

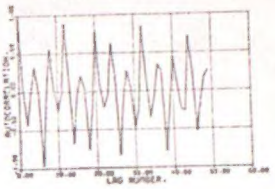
NYQUIST FREQUENCY=1.0HERTZ TIME STEP=0.50.
NO.OF DATA POINTS=64...
NO.OF COEFFICIENTS=50.
NO.OF FREQUENCY POINTS=1000...
AKAIKE FPE SERIES FOR COEFFICIENT ORDER.




b/

MEM. NUMERICAL EXPERIMENT. 1.3

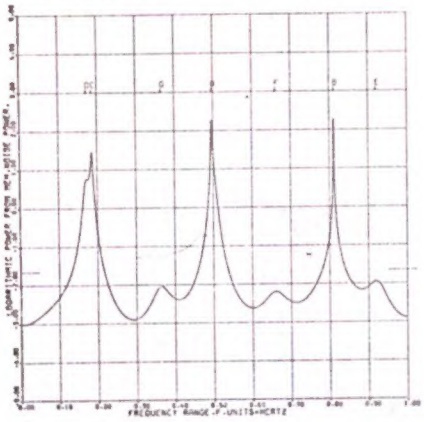
NYQUIST FREQUENCY=1.0HERTZ TIME STEP=0.50.
NO.OF DATA POINTS=64...
NO.OF COEFFICIENTS=50.
NO.OF FREQUENCY POINTS=1000...
AUTOCORRELATION OF TIME SERIES.




c/

MEM. NUMERICAL EXPERIMENT. 1.3

NYQUIST FREQUENCY=1.0HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=64...
NO.OF COEFFICIENTS=21.
NO.OF FREQUENCY POINTS=1000...
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



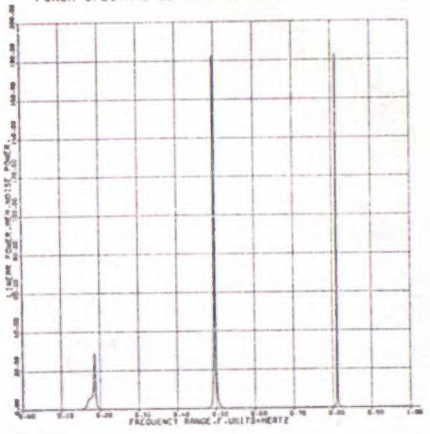

LINE SPECTRUM POWERS.

| NO. | FREQUENCY. | AREA. |
|---|---|---|
| A. | 4.9650E-1 | 1.5911E0 |
| B. | 8.1181E-1 | 6.3519E-1 |
| C. | 1.0619E-1 | 4.4152E-1 |
| D. | 1.7219E-1 | 2.1077E-3 * |
| E. | 9.2096E-1 | 2.2549E-3 * |
| F. | 6.6166E-1 | 1.5950E-3 * |
| G. | 3.6557E-1 | 1.4621E-3 |

d/

MEM. NUMERICAL EXPERIMENT. 1.3

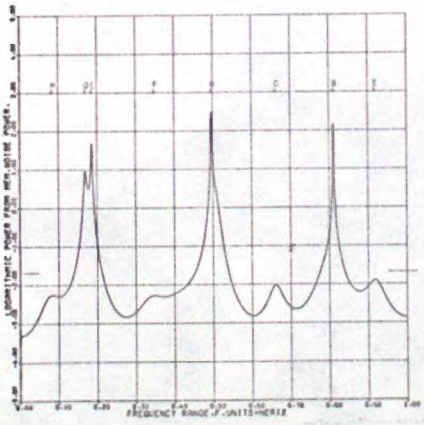
NYQUIST FREQUENCY=1.0HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=64...
NO.OF COEFFICIENTS=21.
NO.OF FREQUENCY POINTS=1000...
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.




e/

MEM. NUMERICAL EXPERIMENT. 1.3

NYQUIST FREQUENCY=1.0HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=64...
NO.OF COEFFICIENTS=24.
NO.OF FREQUENCY POINTS=1000...
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



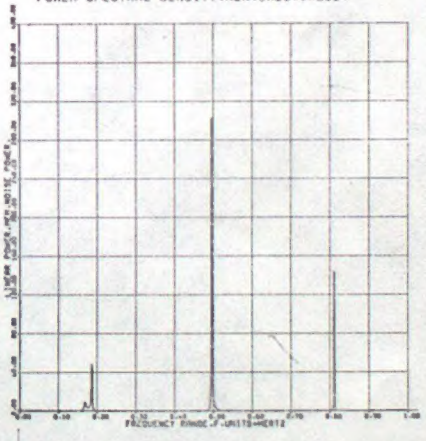

LINE SPECTRUM POWERS.

| NO. | FREQUENCY. | AREA. |
|---|---|---|
| A. | 4.9650E-1 | 1.4945E0 |
| B. | 8.1181E-1 | 5.3259E-1 |
| C. | 1.0619E-1 | 4.0119E-1 |
| D. | 1.6913E-1 | 1.9100E-3 |
| E. | 9.1792E-1 | 2.2507E-3 |
| F. | 5.4835E-1 | 1.3109E-3 * |
| G. | 5.8164E-1 | 1.0599E-3 |
| H. | 8.3053E-2 | 8.9197E-4 * |

f/

MEM. NUMERICAL EXPERIMENT. 1.3

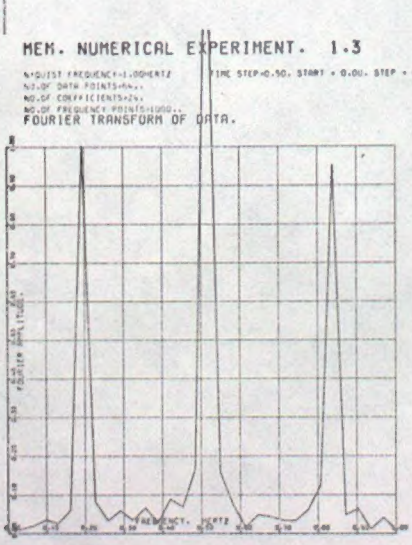
NYQUIST FREQUENCY=1.0HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=64...
NO.OF COEFFICIENTS=24.
NO.OF FREQUENCY POINTS=1000...
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.




g/

MEM. NUMERICAL EXPERIMENT. 1.3

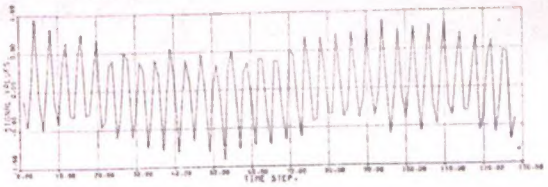
NYQUIST FREQUENCY=1.0HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=64...
NO.OF COEFFICIENTS=24.
NO.OF FREQUENCY POINTS=1000...
FOURIER TRANSFORM OF DATA.




h/

MEM. NUMERICAL EXPERIMENT. 1.5

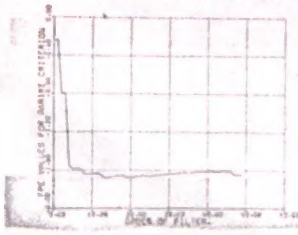
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=60.
NO.OF FREQUENCY POINTS=1000..
TIME SERIES DATA.




a/

MEM. NUMERICAL EXPERIMENT. 1.5

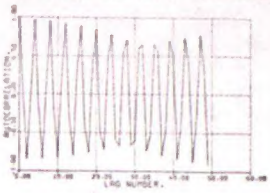
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=60.
NO.OF FREQUENCY POINTS=1000..
AKAIKE FPE SERIES FOR COEFFICIENT ORDER.



b/

MEM. NUMERICAL EXPERIMENT. 1.5

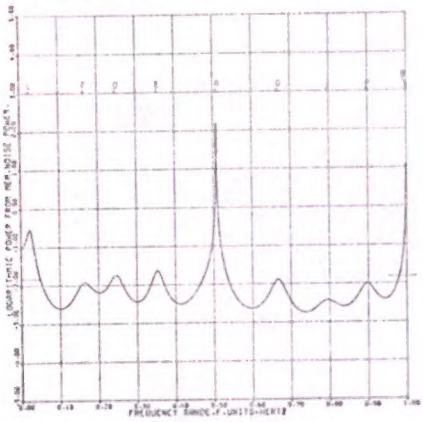
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=60.
NO.OF FREQUENCY POINTS=1000..
AUTOCORRELATION OF TIME SERIES.




c/

MEM. NUMERICAL EXPERIMENT. 1.5

NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=26.
NO.OF FREQUENCY POINTS=1000..
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



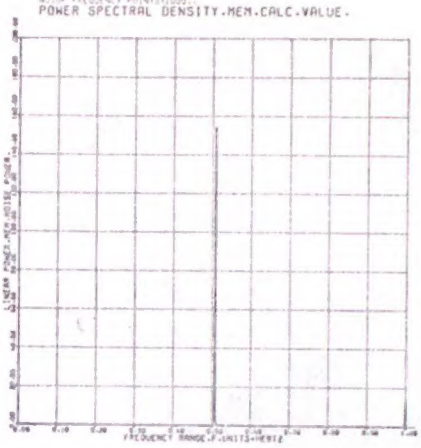

LINE SPECTRUM POWERS.

| NO. | FREQUENCY. | AMER. | |
|---|---|---|---|
| A. | 5.0051E-1 | 5.9044E-1 | |
| B. | 1.0000E0 | 7.1547E-2 | * |
| C. | 2.2022E-2 | 1.6542E-2 | |
| D. | 1.4622E-1 | 2.3670E-3 | |
| E. | 5.5455E-1 | 2.1804E-3 | |
| F. | 1.6619E-1 | 2.1597E-3 | * |
| G. | 6.6767E-1 | 1.6167E-3 | |
| H. | 8.9780E-1 | 1.5660E-3 | |
| I. | 7.9500E-1 | 1.2431E-3 | * |

d/

MEM. NUMERICAL EXPERIMENT. 1.5

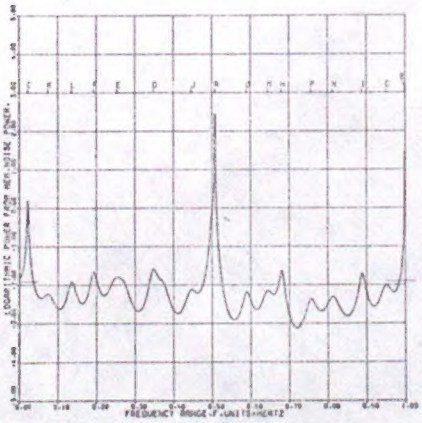
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=26.
NO.OF FREQUENCY POINTS=1000..
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.




e/

MEM. NUMERICAL EXPERIMENT. 1.5

NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=42.
NO.OF FREQUENCY POINTS=1000..
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



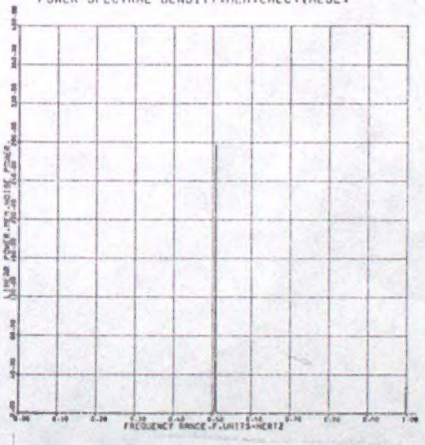

LINE SPECTRUM POWERS.

| NO. | FREQUENCY. | AMER. | |
|---|---|---|---|
| A. | 5.0051E-1 | 9.2124E-1 | |
| B. | 1.0000E0 | 1.2730E-1 | * |
| C. | 2.2022E-2 | 1.5286E-2 | |
| D. | 5.4835E-1 | 2.4496E-3 | |
| E. | 2.5425E-1 | 2.2556E-3 | |
| F. | 1.9519E-1 | 1.2526E-3 | |
| G. | 9.5195E-1 | 1.1300E-3 | * |
| H. | 6.8068E-1 | 9.5724E-4 | |
| I. | 8.0815E-1 | 9.0615E-4 | |
| J. | 4.4745E-1 | 8.5619E-4 | * |
| K. | 7.4074E-2 | 8.2139E-4 | * |
| L. | 1.5614E-1 | 8.1548E-4 | |
| M. | 6.4455E-1 | 7.2780E-4 | * |
| N. | 8.7581E-1 | 5.8156E-4 | |
| O. | 5.5059E-1 | 4.7691E-4 | |
| P. | 3.5776E-1 | 4.5030E-4 | |

f/

MEM. NUMERICAL EXPERIMENT. 1.5

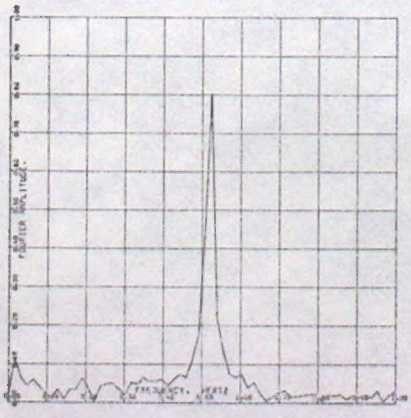
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=42.
NO.OF FREQUENCY POINTS=1000..
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.




g/

MEM. NUMERICAL EXPERIMENT. 1.5

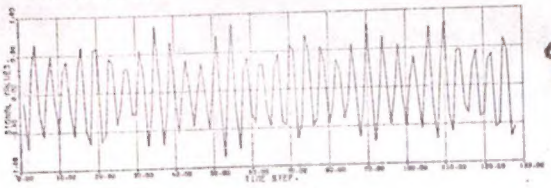
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=42.
NO.OF FREQUENCY POINTS=1000..
FOURIER TRANSFORM OF DATA.




h/

MEM. NUMERICAL EXPERIMENT. 1.6

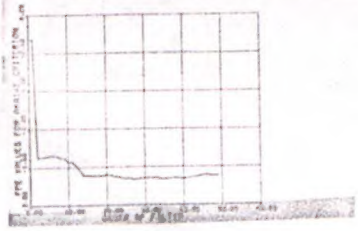
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=50.
NO.OF FREQUENCY POINTS=1000..
TIME SERIES DATA.



a/

MEM. NUMERICAL EXPERIMENT. 1.6

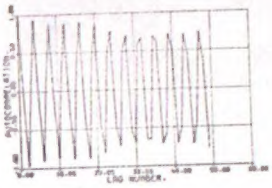
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=50.
NO.OF FREQUENCY POINTS=1000..
AKAIKE FPE SERIES FOR COEFFICIENT ORDER.



b/

MEM. NUMERICAL EXPERIMENT. 1.6

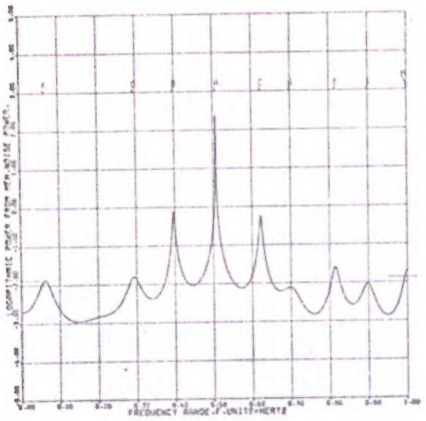
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=50.
NO.OF FREQUENCY POINTS=1000..
AUTOCORRELATION OF TIME SERIES.



c/

MEM. NUMERICAL EXPERIMENT. 1.6

NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=24.
NO.OF FREQUENCY POINTS=1000..
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



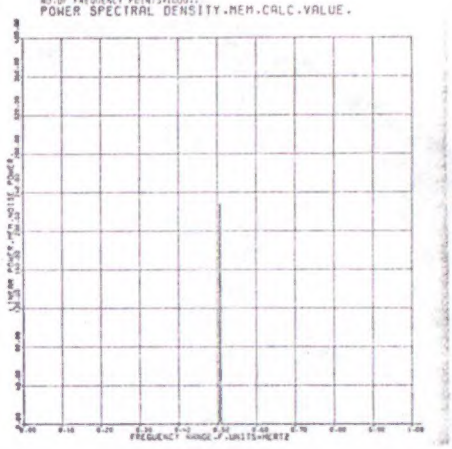
LINE SPECTRUM POWERS.

| NO. | FREQUENCY. | AREA. | |
|---|---|---|---|
| A. | 5.0751E-1 | 7.5875E-1 | |
| B. | 1.9740E-1 | 1.4532E-2 | |
| C. | 6.2182E-1 | 1.2264E-2 | |
| D. | 2.9522E-1 | 1.7359E-3 | |
| E. | 8.1582E-1 | 1.4542E-3 | |
| F. | 6.2062E-2 | 1.3869E-3 | |
| G. | 6.9830E-1 | 1.3667E-3 | * |
| H. | 1.0000E0 | 1.0429E-3 | * |
| I. | 8.5895E-1 | 9.1940E-4 | |

d/

MEM. NUMERICAL EXPERIMENT. 1.6

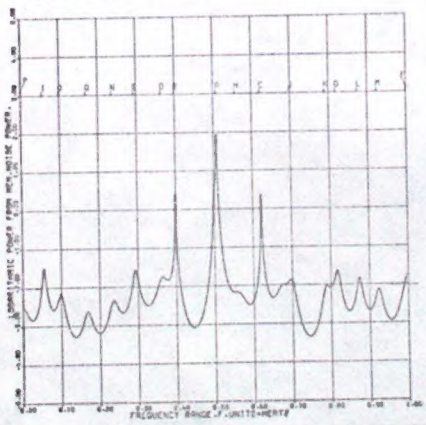
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=24.
NO.OF FREQUENCY POINTS=1000..
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.



e/

MEM. NUMERICAL EXPERIMENT. 1.6

NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=42.
NO.OF FREQUENCY POINTS=1000..
POWER SPECTRAL DENSITY.LOG.CALC.VALUE.



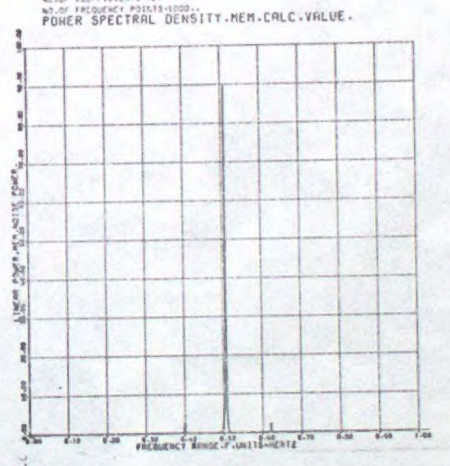
LINE SPECTRUM POWERS.

| NO. | FREQUENCY. | AREA. | |
|---|---|---|---|
| A. | 5.0851E-1 | 5.5569E-1 | |
| B. | 5.9940E-1 | 1.4611E-2 | |
| C. | 6.2182E-1 | 1.2264E-2 | |
| D. | 5.6451E-1 | 1.7515E-3 | * |
| E. | 2.4522E-1 | 1.2046E-3 | |
| F. | 1.0000E0 | 1.1646E-3 | * |
| G. | 8.1782E-1 | 1.1381E-3 | |
| H. | 5.5715E-1 | 9.4571E-4 | * |
| I. | 5.5255E-2 | 8.8641E-4 | * |
| J. | 6.9770E-1 | 8.3712E-4 | * |
| K. | 7.8837E-1 | 6.5745E-4 | * |
| L. | 8.3548E-1 | 6.5119E-4 | |
| M. | 9.2492E-1 | 4.6651E-4 | |
| N. | 2.7624E-1 | 4.1787E-4 | |
| O. | 9.6099E-2 | 4.0265E-4 | |
| P. | 0.0000E0 | 3.7544E-4 | * |
| Q. | 1.6491E-1 | 3.7630E-4 | |

f/

MEM. NUMERICAL EXPERIMENT. 1.6

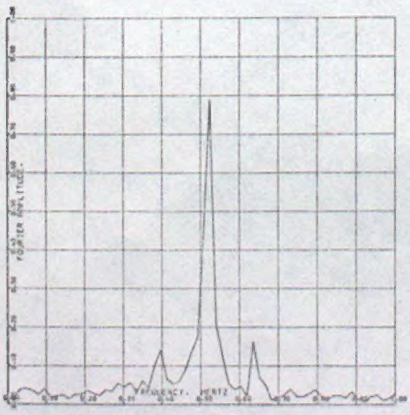
NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=42.
NO.OF FREQUENCY POINTS=1000..
POWER SPECTRAL DENSITY.MEM.CALC.VALUE.



g/

MEM. NUMERICAL EXPERIMENT. 1.6

NYQUIST FREQUENCY=1.00HERTZ TIME STEP=0.50. START = 0.00. STEP = 0.00.
NO.OF DATA POINTS=128..
NO.OF COEFFICIENTS=42.
NO.OF FREQUENCY POINTS=1000..
FOURIER TRANSFORM OF DATA.



h/